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A SOLUTION OF OVSIANNIKOV'S REDUCTION PROBLEM

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Abstract: In his book "Group Analysis of Differential Equations", Ovsiannikov poses a problem called the problem of reduction. Roughly speaking, it asks for necessary and sufficient conditions under which symmetry reduction, in a certain sense, is possible. When it *is*, the construction of non-invariant solutions — as well as of invariant ones — of partial differential equations (PDEs) becomes easier. Thus a solution of the reduction problem would be useful. A solution, under certain assumptions that appear to be natural, is presented in this paper. The method used indicates not only *when* the reduction is possible but also *how* it can be carried out.

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1. Introduction

The purpose of this paper is to present a solution of the reduction problem (Ovsiannikov [12]) of Ovsiannikov under certain conditions that appear to be natural. To state the problem, let $\Delta = 0$ be a system of partial differential equations (PDEs) for q unknowns $u_1, u_2, \ldots, u_q$ in p independent variables $x_1, x_2, \ldots, x_p$. We shall write $x = (x_1, x_2, \ldots, x_p), u = (u_1, u_2, \ldots, u_q)$. Let G be the full local Lie group of point symmetries of the system. Thus G acts on the manifold $M = \mathbf{R}^p \times \mathbf{R}^q$. Throughout this paper, all objects — manifolds, mappings, vector fields — will be assumed smooth. Also, all group actions considered will be local and semiregular, i.e., for all points z in M on which G acts, the orbits $Gz = \{gz | g \in G\}$ are submanifolds of M of locally constant dimension. Let Γ be a solution manifold, defined implicitly by $F(x, u) = 0$, where $F : \mathbf{R}^p \times \mathbf{R}^q \to \mathbf{R}^q$, $F(x, u) = (F^1(x, u), \ldots, F^q(x, u))$, is assumed to be of full rank. Let $H \subset G$ be a Lie subgroup. Let $\mathcal{U} \subset \Gamma$ be an open set small enough that the submanifolds $Hz, z \in \mathcal{U}$, are all of the same dimension. Since the group actions are local, we shall rename $\mathcal{U}$ as Γ. Ovsiannikov makes the assumption that $H\Gamma$ (the union of all the orbits Hz, as z varies over Γ) is itself a manifold and calls it the orbit manifold. By definition, then, $H\Gamma$ is invariant under H, but in general Γ is not invariant under H. When Γ *is* invariant under H, the corresponding solution is called an invariant (or H-invariant) solution and Γ an invariant manifold. Given H and Γ, Ovsiannikov defines two quantities, namely the defect δ and the rank ρ as follows:

$$\delta \equiv \delta(\Gamma, H) = \dim(H\Gamma) - \dim \Gamma$$

and

$$\rho \equiv \rho(\Gamma, H) = \dim(H\Gamma) - \dim Hz, \quad z \in \Gamma.$$

The defect δ measures the extent to which Γ fails to be invariant under H. The rank ρ is the local codimension in $H\Gamma$ of the orbit of a single point in Γ, and this is equal to the maximal number (Olver [8], Ovsiannikov [12]) of functionally independent H-invariant functions on $H\Gamma$. (Any smooth function defined on $H\Gamma$ and invariant under H can be expressed as a function of ρ functionally independent invariant functions.) Ovsiannikov [12] has shown that if $H' \subset H$ is a Lie subgroup and if $\delta' = \delta(\Gamma, H')$ and $\rho' = \rho(\Gamma, H')$ then $\delta' \leq \delta$ and $\rho' \geq \rho$. From a practical point of view, invariant solutions — corresponding to $\delta = 0$ —

are the easiest ones to construct: one can use Lie's algorithm, as refined by Ovsiannikov [12] and Olver [8]. If a solution is not invariant, the best one can hope for is reducing the defect while keeping the rank the same. This is what is meant by reduction in the sense of Ovsiannikov. We can now state his

Reduction Problem: Given H and Γ as defined above, what are the necessary and sufficient conditions for the existence of a Lie subgroup $H' \subset H$ s.t. $\delta' < \delta$ and $\rho' = \rho$?

Before we state the assumptions under which the problem can be solved, a couple of remarks are in order:

1. Reduction in the sense of Ovsiannikov is different from reduction or reducibility as defined by one of us in earlier work (Sastri et al [16, 18]) and as commonly understood in the literature. There, a solution is called irreducible if there is no non-trivial subgroup of G under which it is invariant. The difference is that in Ovsiannikov's case, control over *two* parameters — the rank and the defect — is sought, whereas in our earlier definition, only one parameter, the defect, matters. Therefore, to make the terminology compatible, we let "rank-invariant reduction" mean reduction in the sense of Ovsiannikov. Note further that while Ovsiannikov's definition is useful in determining whether a given solution is reducible, our earlier definition is related to the question of whether a given system of PDEs has irreducible solutions with respect to G. For an example, see Quinn et al [14].

2. Ovsiannikov introduces the notion of a partially invariant solution (Ovsiannikov [12]): Γ is partially invariant if its defect is strictly positive and if $H\Gamma$ is not the whole of M. For our purposes, what is of interest is a measure of the lack of invariance of a solution, namely the defect, whatever it happens to be.

There has been considerable work on partial invariance (Bytev [1], Ibragimov [3, 4], Martina et al [5, 6], Menćšikov [7], Ondich [9], Ovsiannikov [11-13], Quinn et al [14], Rai and Sastri [15], Sastri et al [16-18], Strampp [19]), but as far as we know, Ovsiannikov's reduction problem has not been addressed. Now, in order to state the conditions under which we solve the problem, some notation is needed. Let $\boldsymbol{g}, \boldsymbol{h}$ be the Lie algebras of G and H, respectively. If $\boldsymbol{w}$ is a vector field in $\boldsymbol{h}$, let

$\boldsymbol{h}(z) = \{\boldsymbol{w}(z) | \boldsymbol{w} \in \boldsymbol{h}\}$, $z \in \Gamma$. Then $\boldsymbol{h}(z)$ is a vector space; even if $\boldsymbol{h}$ is infinite dimensional, $\boldsymbol{h}(z)$ is finite dimensional, since it is a subspace of $TM|_z$, the tangent space of M at z, and $\dim(TM|_z) = p + q$. Let $H(\Gamma)$ be the maximal Lie subgroup of H which leaves Γ invariant. Clearly, such a subgroup exists. For if Γ is invariant under H, $H(\Gamma) = H$, and if the only subgroup of H under which Γ is invariant is the trivial subgroup, then $H(\Gamma)$ is the trivial subgroup. Otherwise, $H(\Gamma)$ is a proper subgroup of H. That it is a Lie group follows (Olver [8], Ovsiannikov [12]) from the fact that if a manifold is invariant under the action of two vector fields, then it is also invariant under the action of their Lie bracket. In what follows, we shall adopt the notation $\boldsymbol{a}\langle \boldsymbol{w}_1, \boldsymbol{w}_2, \ldots \rangle$ to denote the Lie algebra with basis $\{\boldsymbol{w}_1, \boldsymbol{w}_2, \ldots\}$ and $\langle \boldsymbol{w}_1, \boldsymbol{w}_2, \ldots \rangle$ to denote the corresponding vector space.

We are now ready to state the assumptions:

(i) $H\Gamma$ is a manifold.

(ii) For $z \in \Gamma$, $\dim(T(H\Gamma)|_z) = \dim\langle \boldsymbol{h}(z), T\Gamma|_z \rangle$, where $T\Gamma|_z$ and $T(H\Gamma)|_z$ are the tangent spaces, respectively, of Γ and $H\Gamma$, at z, and $\langle \boldsymbol{h}(z), T\Gamma|_z \rangle$ is the span of $\boldsymbol{h}(z)$ and $T\Gamma|_z$.

(iii) The action of H "splits along Γ", i.e.,

$$\boldsymbol{h}(\Gamma)(z) = \boldsymbol{h}(z) \cap T\Gamma|_z, \qquad z \in \Gamma,$$

where $\boldsymbol{h}(\Gamma)$ is the Lie algebra of $H(\Gamma)$.

We note that Ovsiannikov himself makes assumption (i); the other two are new. To see that these assumptions are not always satisfied, let's consider the following examples:

1. Let Γ be the x-axis in $\mathbf{R}^2(x,y)$, $\boldsymbol{w} = \partial_x + 2x\partial_y$, and $\boldsymbol{h} = \boldsymbol{a}\langle \boldsymbol{w} \rangle$. Near $(0,0)$, (i) $H\Gamma$ is not a manifold, (ii) $\dim\langle \boldsymbol{h}(z), T\Gamma|_z \rangle$ is not locally constant and (iii) $\boldsymbol{h}(\Gamma)$ is the zero algebra, so that $\boldsymbol{h}(\Gamma)(0,0)$ is the zero vector space, but $\boldsymbol{h}(0,0) \cap T\Gamma|_{(0,0)} = \langle (1,0) \rangle$, so the action of H does not split along Γ.

2. Let Γ be the x-axis in $\mathbf{R}^2(x,y)$ and $\boldsymbol{h} = \boldsymbol{a}\langle \boldsymbol{w}_1, \boldsymbol{w}_2 \rangle$, where $\boldsymbol{w}_1 = \partial_x$, $\boldsymbol{w}_2 = \partial_y$.
All three assumptions are satisfied.

3. Let Γ be the x-axis in $\mathbf{R}^2(x, y)$ and $\boldsymbol{h} = \boldsymbol{a}\langle \boldsymbol{w}_1, \boldsymbol{w}_2\rangle$, where $\boldsymbol{w}_1 = \partial_y$ and $\boldsymbol{w}_2 = \partial_x + 2x\partial_y$. Assumptions (i) and (ii) are satisfied, but not (iii). For $\boldsymbol{h}(\Gamma)$ is the zero algebra, so that

$$\boldsymbol{h}(0,0) \cap T\Gamma|_{(0,0)} = \langle (1,0)\rangle \neq \boldsymbol{h}(\Gamma)(0,0).$$

2. The Reduction Problem

In what follows, given $z \in \Gamma$, we mean by proj the projection onto the orthogonal complement of $T\Gamma|_z$ in $TM|_z$. We now need a result of Ovsiannikov (see Proposition 2.2 below), which we shall state with a new proof. Our proof has the advantage of suggesting a solution to the reduction problem. We start with

Lemma 2.1. *Let $\boldsymbol{w}_1, \boldsymbol{w}_2, \ldots \boldsymbol{w}_r$ be elements of $\boldsymbol{h}$ s.t. $\forall z \in \Gamma$, $\{\boldsymbol{w}_1(z), \ldots, \boldsymbol{w}_r(z)\}$ is a basis for $\boldsymbol{h}(z)$. Let $Q = (\{\boldsymbol{w}_j(F^i)\})$. Then rank $Q(z) = \dim proj\ \boldsymbol{h}(z)$.*

Proof. Let $\lambda_1, \lambda_2, \ldots, \lambda_r \in \mathbf{R}$ be arbitrary, and let $\boldsymbol{w} = \lambda_1\boldsymbol{w}_1 + \cdots + \lambda_r\boldsymbol{w}_r$. Then, at z

$$Q(\{\boldsymbol{w}_j(z)\})\begin{pmatrix}\lambda_1\\ \vdots\\ \lambda_r\end{pmatrix} = \begin{pmatrix}(\boldsymbol{w}_1F^1)(z)\cdots(\boldsymbol{w}_rF^1)(z)\\ \cdots\cdots\cdots\cdots\cdots\cdots\\ (\boldsymbol{w}_1F^q)(z)\cdots(\boldsymbol{w}_rF^q)(z)\end{pmatrix}\begin{pmatrix}\lambda_1\\ \vdots\\ \lambda_r\end{pmatrix} = \begin{pmatrix}(\boldsymbol{w}\cdot\nabla F^1)(z)\\ \cdots\cdots\\ \cdots\cdots\\ (\boldsymbol{w}\cdot\nabla F^q)(z)\end{pmatrix}$$

so that $Q(\{\boldsymbol{w}_j(z)\})\begin{pmatrix}\lambda_1\\ \vdots\\ \lambda_r\end{pmatrix} = 0$ if $\boldsymbol{w}(z)$ is orthogonal to $\nabla F^i(z)$, $i = 1, \ldots, q$, i.e., $\boldsymbol{w}(z) \in T\Gamma|_z$, since F is of full rank.

Hence $\dim\ker Q(z) = \dim\ker \mathrm{proj}|_{\boldsymbol{h}(z)}$. But

$$\dim\left(\left\{\begin{pmatrix}\lambda_1\\ \vdots\\ \lambda_r\end{pmatrix}\right\}, \lambda_i \in \mathbf{R}\right) = \dim \boldsymbol{h}(z) = r$$

by hypothesis.

Hence, by the rank-nullity theorem,

$$\begin{aligned} \dim \text{range } (Q(z)) &= \dim \text{range proj}|_{\boldsymbol{h}(z)} \\ &= \dim \text{proj } \boldsymbol{h}(z). \end{aligned} \tag{1}$$

Proposition 2.2 (Ovsiannikov) *$\delta = rank\ Q(z)$.*

Proof. From the definition of defect, we have

$$\begin{aligned} \delta &= \dim H\Gamma - \dim \Gamma \\ &= \dim T(H\Gamma)|_z - \dim T\Gamma|_z \\ &= \dim(\boldsymbol{h}(z)) + \dim T\Gamma|_z - \dim(\boldsymbol{h}(z) \cap T\Gamma|_z) - \dim T\Gamma|_z \\ &= \dim(\boldsymbol{h}(z)) - \dim(\boldsymbol{h}(z) \cap T\Gamma|_z) \\ &= \dim \text{proj } \boldsymbol{h}(z) \\ &= \text{rank } Q(z). \end{aligned}$$

Corollary 2.3. $\delta = \dim \boldsymbol{h}(z) - \dim \boldsymbol{h}(\Gamma)(z)$.

This follows directly from assumption (iii) and the proof of Prop. 2.2.

We recall (Olver [8], Ovsiannikov [12]) that by the existence and uniqueness theorem, a vector field $\mathbf{v}$ leaves a submanifold Γ invariant if $\mathbf{v}(z) \in T\Gamma|_z$ for all $z \in \Gamma$. So given H and Γ it is straightforward to find $H(\Gamma)$. Suppose that $\{\boldsymbol{w}_\alpha\}$ is a basis for the possibly infinite dimensional Lie algebra $\boldsymbol{h}$. Then $\boldsymbol{w} = \sum_{j=1}^{k} \lambda_{\alpha_j} \boldsymbol{w}_{\alpha_j} \in \boldsymbol{h}(\Gamma) \iff \boldsymbol{w}(F^i) = \sum_{j=1}^{k} \lambda_{\alpha_j} \boldsymbol{w}_{\alpha_j}(F^i) = 0$ along Γ, $i = 1, \ldots, q \iff (\lambda_{\alpha_1}, \ldots, \lambda_{\alpha_k})$ is a constant vector in $\ker(\boldsymbol{w}_{\alpha_j}(F^i))(z)$ for all $z \in \Gamma$.

Proposition 2.4. $\rho(\Gamma, H) = \rho(\Gamma, H(\Gamma))$.

Proof. We have, by definition

$$\begin{aligned} \rho(\Gamma, H) &= \dim(H\Gamma) - \dim Hz \quad (z \in \Gamma) \\ &= \dim T(H\Gamma)|_z - \dim Hz \\ &= \dim T(H\Gamma)|_z - \dim \boldsymbol{h}(z) \\ &= \dim T\Gamma|_z + \delta - \dim \boldsymbol{h}(z) \\ &= \dim T\Gamma|_z - \dim \boldsymbol{h}(\Gamma)(z) + \delta - (\dim \boldsymbol{h}(z) - \dim \boldsymbol{h}(\Gamma)(z)) \\ &= \dim T\Gamma|_z - \dim \boldsymbol{h}(\Gamma)(z) \\ &= \rho(\Gamma, H(\Gamma)) \end{aligned}$$

by semiregularity.

We next state a result of Ovsiannikov's with a new, and simpler, proof.

Lemma 2.5. *Let $H' \subset H$ be a Lie subgroup. Then (i) $\delta' \leq \delta$ and (ii) $\rho' \geq \rho$.*

Proof.

$$(i)\ \ \delta' = \dim \text{proj}\ \boldsymbol{h}'(z) \leq \dim \text{proj}\ \boldsymbol{h}(z) = \delta$$

and

$$\begin{aligned} (ii)\ \ \rho' &= \dim T\Gamma|_z - \dim \boldsymbol{h}'(\Gamma)(z) \\ &\geq \dim T\Gamma|_z - \dim \boldsymbol{h}(\Gamma)(z) \\ &= \rho. \end{aligned}$$

The solution to Ovsiannikov's reduction problem follows easily now.

Corollary 2.6. *$H' \subset H$ reduces (Γ, H) in the sense of Ovsiannikov iff (i) $\delta' = \dim \text{proj}\ \boldsymbol{h}'(z) < \dim \text{proj}\ \boldsymbol{h}(z) = \delta$, and (ii) $\boldsymbol{h}'(\Gamma)(z) = \boldsymbol{h}(\Gamma)(z)$.*

We observe that since

$$\boldsymbol{h}'(\Gamma)(z) \subset \boldsymbol{h}(\Gamma)(z), \quad \dim \boldsymbol{h}'(\Gamma)(z) = \dim \boldsymbol{h}(\Gamma)(z)$$

iff $\boldsymbol{h}'(\Gamma)(z) = \boldsymbol{h}(\Gamma)(z)$.

For completeness, we include a reformulation of the Ovsiannikov (rank-invariant) reduction problem. In an immediate sense, this will only modestly extend the results so far, but will also correspond to an analogous problem raised in Sastri et al [16, 18]. The basic observation is that the defect δ and the rank ρ are numerical quantities. In Ovsiannikov's work, minimizing δ and ρ corresponds to minimizing the complexity of the algorithm and maximizing the invariance of possible solutions. Strictly speaking then, it is not necessary to restrict ourselves to subgroups of a given H. The objective can simply be to minimize δ and ρ. So, as in Sastri et al [16, 18], we extend the context to the full Lie symmetry group G. The extended problem is then to determine the existence of a rank-invariant reduction in G, that is, to find $H' \subset G$ (H' not necessarily a subgroup of H) such that $\delta' < \delta$ and $\rho' = \rho$. Again, assuming that actions split along Γ we obtain the following:

Corollary 2.7. *$H' \subset G$ gives a rank-invariant reduction of (Γ, H) if:*

(i) $\dim\, proj\, \boldsymbol{h}'(z) < \dim\, proj\, \boldsymbol{h}(z)$ *and*

(ii) $\dim \boldsymbol{h}'(\Gamma)(z) = \dim \boldsymbol{h}(\Gamma)(z)$.

We observe that it is possible to have a chain of reductions $H^{(k)} \subset \cdots H^{(2)} \subset H^{(1)} \subset H$, but the best possible reduction is obtained by choosing $H' = H(\Gamma)$. Then, since Γ is invariant under $H(\Gamma)$, $\delta' = 0$ and, by Prop. 2.4, $\rho' = \rho$. In other words, if $\boldsymbol{h}(\Gamma) \neq \{0\}$, then there is a rank-invariant reduction. The converse is not true in general. (See Examples 3 and 4 below.)

2. Examples

We remark that all three of our assumptions are satisfied by the examples considered by Ovsiannikov.

1. This example is due to Ovsiannikov ([12], p.286 and p.293), although the analysis given here is different from his.

   Consider the equations of transonic flow:

   $$u_y = v_x, \qquad v_y = -uu_x, \qquad x, y \in \mathbf{R}.$$

   This is a nonlinear system whose full Lie symmetry group is infinite dimensional and acts on $M = \mathbf{R}^4(x, y, u, v)$. Let H be the two-dimensional subgroup of this group with generators $w_1 = x\partial_x + y\partial_y$ and $w_2 = \partial_v$. If $z = (x, y, u, v)$ and $\lambda = \frac{x}{y}$, then $F(z) = (F_1(z), F_2(z)) = (u - f(\lambda), v - g(\lambda) + c\, ln\, y)$, where f and g are some functions and c is a constant. Γ is defined as the zero-set of F, i.e., it is given by $u = f(\lambda), v = g(\lambda) - c\, ln\, y$. So $\dim \Gamma = 2$. Now we have $Q(z) = \begin{pmatrix} 0 & 0 \\ c & 1 \end{pmatrix}$, so that $\delta = \text{rank } Q(z) = 1$. The kernel of $Q(z)$ consists of vectors of the form $\begin{pmatrix} \alpha_1 \\ \alpha_2 \end{pmatrix}$ where $\alpha_2 = -c\alpha_1$. Choosing $\alpha_1 = 1$, we find that $\boldsymbol{h}(\Gamma)$ is the algebra generated by $\boldsymbol{w} = x\partial_x + y\partial_y - c\partial_v$ and so $\rho = \dim\Gamma - \dim \boldsymbol{h}(\Gamma) = 1$. Ovsiannikov chooses $\boldsymbol{h}'$ to be the algebra generated by $x\partial_x + y\partial_y - c\partial_v$, i.e., s.t. $\boldsymbol{h}' = \boldsymbol{h}(\Gamma)$. The conditions of Corollary 2.6 are satisfied, and rank-invariant reduction takes place.

2. This example is also due to Ovsiannikov ([12], p.279), although, again, the analysis given here is different from his. The ambient space is $M = \mathbf{R}^4(x, y, z, v)$; if $\boldsymbol{z} = (x, y, z, v)$, $F(\boldsymbol{z}) = (F_1(\boldsymbol{z}), F_2(\boldsymbol{z})) = (y - \lambda xz, v)$, and Γ is defined as the zero-set of F. Thus Γ is given by $y - \lambda xz = 0$, $v = 0$. Clearly, $\dim\Gamma = 2$. Let $\boldsymbol{h}$ be the algebra generated by $\boldsymbol{w}_1 = x\partial_y$ and $\boldsymbol{w}_2 = \partial_z$. It is easy to see that $\dim(H\Gamma) = 3$ and $\dim(H\boldsymbol{z}) = 2$, so that $\rho = 1$. We have $Q(\boldsymbol{z}) = \begin{pmatrix} x & -\lambda x \\ 0 & 0 \end{pmatrix}$, so that if $x \neq 0$, rank $Q(\boldsymbol{z}) = 1 = \delta$, the defect.

   The kernel of $Q(\boldsymbol{z})$ consists of all vectors $\begin{pmatrix} \alpha_1 \\ \alpha_2 \end{pmatrix}$ of the form $\alpha_1 = \lambda\alpha_2$, i.e., $\alpha_2 \begin{pmatrix} \lambda \\ 1 \end{pmatrix}$, $\alpha_2 \neq 0$.

   Setting $\alpha_2 = 1$, we find that $\boldsymbol{w}(\boldsymbol{z}) = (\lambda x\partial_y + \partial_z)(\boldsymbol{z})$ is in $T\Gamma|_{\boldsymbol{z}}$, i.e., $\boldsymbol{w}$ is the generator of $\boldsymbol{h}(\Gamma)$. Ovsiannikov chooses $\boldsymbol{h}'$ to be the algebra generated by $\lambda x\partial_y + \partial_v$, i.e., s.t. $\boldsymbol{h}' = \boldsymbol{h}(\Gamma)$. The conditions of Corollary (2.6) are satisfied, since

$$\begin{aligned} \delta' &= \text{dim proj } \boldsymbol{h}'(\boldsymbol{z}) = \text{dim proj } \boldsymbol{h}(\Gamma)(\boldsymbol{z}) = 0 < \delta = 1 \\ &\text{and} \quad \boldsymbol{h}'(\Gamma)(\boldsymbol{z}) = \boldsymbol{h}(\Gamma)(\boldsymbol{z}), \end{aligned}$$

   and hence rank-invariant reduction occurs.

3. Suppose that Γ is the x-axis in $\mathbf{R}^3(x, y, z)$ and let $\mathbf{e}_2 = (0, 1, 0)$, $\mathbf{e}_3 = (0, 0, 1)$ and let $\boldsymbol{h} = \boldsymbol{a}\langle \mathbf{e}_2, \mathbf{e}_3\rangle$. Then $\delta(\Gamma, H) = 2$ and $\rho(\Gamma, H) = 1$. Let $\boldsymbol{w}$ be any non-zero vector field in $\boldsymbol{h}$ and let $\boldsymbol{h}' = \boldsymbol{a}\langle \boldsymbol{w}\rangle$. Then $\delta(\Gamma, H') = 1$ while the rank remains $\rho(\Gamma, H') = \rho(\Gamma, H) = 1$. Note that $\boldsymbol{h}(\Gamma) = \boldsymbol{h}'(\Gamma) = \{0\}$.

   In our next example, we show that the basic situation given in Example 3 occurs in the system of partial differential equations governing transonic flow.

4. As in Example 1, with $w = x\partial_x + y\partial_y - c\partial_v$, $h' = a\langle w\rangle$ gives a rank-invariant reduction. But Ovsiannikov's algorithm can be used again, this time to obtain solutions partially invariant with respect to $h' = a\langle w\rangle$. Let $z = (x, y, u, v)$. For arbitrary c, an extended set of solutions (Sastri [17]) partially invariant with respect to H' includes functions which satisfy

$$\begin{aligned} u_y &= \mp(-u)^{\frac{1}{2}} u_x \\ v &= k(\lambda, u) - c\, ln\, y, \quad \lambda = \frac{x}{y}, \quad y \neq 0, \end{aligned}$$

where k satisfies a derived equation. In particular, we may suppose $k_\lambda = 0$. Now with $h = a\langle w_1, w_2\rangle$ and Γ one of these further solutions, we obtain

$$Q(z) = \begin{bmatrix} -(xu_x + yu_y) & 0 \\ c & 1 \end{bmatrix}.$$

As in Sastri [17], there exist solutions for which $xu_x + yu_y \neq 0$, so that $rankQ(z) = \delta = 2$. And because $h(\Gamma) = \{0\}$, we also get $\rho = 2$. Hence, much as in Example 3 above, if $\mathbf{v}$ is any non-zero vector field in h, let $h'' = a\langle \mathbf{v}\rangle$. Then $h''(\Gamma) = h(\Gamma) = \{0\}$, $\delta'' = 1 < \delta = 2$ and $\rho'' = \rho = 2$. Therefore, there exist partially invariant solutions with respect to H' which are rank-invariant reducible with respect to H, but which are not invariant with respect to any subgroup of H.

5. If $\delta > 1$ and $z \in \Gamma$, then there exists $w \in h$ such that proj $w(z) \neq 0$. So the dimension of the linear space $\dim\langle w, z\rangle, h(\Gamma)(z)\rangle = \dim h(\Gamma)(z) + 1 < \dim h(\Gamma) + \delta$. In general however, this may not yield a reduction. For since w and $h(\Gamma)$ may not be in involution, the real linear dimension of the Lie algebra $a\langle w, h(\Gamma)\rangle(z)$ may also be larger than $\dim h(\Gamma)(z) + 1$.

3. Concluding Remarks

Local to a point $z \in \Gamma$, a given group of symmetries may not "split" along Γ. In such cases, $H\Gamma$ may fail to be a manifold. We may nevertheless enquire into the orbit structure. One way to do this would be to investigate general group actions on sets of submanifolds. This is independent of differential equations as such and becomes a geometric question that could apply, for example, to confoliations (Eliashberg and Thurston [2]) and moduli spaces. Other applications could be in "spontaneous symmetry breaking (SSB)" in particle physics. An SSB can be characterized by the orbit (under an "internal" group action) of a function which minimizes a potential (O'Raifeartaigh [10]). One feature of a generalized theory for orbits of manifolds could be an appropriately generalized definition of dimension of a subset of a manifold. For instance, one could take the supremum, over a neighborhood, of dimension in the usual sense. In that case we obtain $\delta \geq$ rank $Q(z)$, where strict inequality may occur. Preliminary results indicate, however, that a more refined analysis is needed. As is well known, "dimension" can be defined in many ways, depending on the setting. In the context of orbits $H\Gamma$, certain topological definitions promise to be of use. But this would take us beyond the scope of the present paper; we shall pursue these matters elsewhere.

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