Communication in Applied Analysis 4 (2000), no. 4, 475-480

REMARKS ON PARTIALLY INVARIANT SOLUTIONS OF DIFFERENTIAL EQUATIONS

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Communicated by A. Dishliev

ABSTRACT: A solution of a system of PDEs is said to be partially invariant w.r.t. a subgroup H of the group G of Lie symmetries of the system, if the solution manifold is not necessarily invariant under H but is invariant under a subgroup $H' \leq H$. In this paper, a necessary and sufficient condition is given for a solution to be partially invariant.

AMS (MOS) Subject Classification. 58G35

1. INTRODUCTION

Let $\Delta = 0$ represent a system of differential equations. For example $\Delta = u_{xx} - u_t$ gives the classical (partial differential) heat equation. Let $G = G(\Delta)$ be the full Lie symmetry group of the equations $\Delta = 0$, i.e. the group of transformations of the base manifold $Z = X \times U$ (where X represents the space of independent variables and U the space of dependent variables) which leaves the equations invariant (and so solutions get mapped to solutions Olver [5]). Lie's fundamental work dates from the late 19th century. One of his results was an algorithm by which to obtain solutions invariant under a prescribed subgroup $H \leq G(\Delta)$ (particularly when H is a 1-parameter subgroup). In the 1960's, Ovsiannikov [7] extended Lie's algorithm by introducing the notion of "partial invariance". The extended algorithm generates solutions which are not necessarily invariant with respect to the given symmetry group $G(\Delta)$.

Received November 19, 1997

2. PARTIAL INVARIANCE

All objects referred to in this paper are defined in the usual way. See, for example Helgason [2], Singer and Sternberg [8], Olver [5]. Typically G, H will be Lie groups, possibly infinite dimensional; $\mathbf{g}, \mathbf{h}$ will be their respective Lie algebras. A group G will act locally on a connected manifold $Z = X \times U = \mathbb{R}^p \times \mathbb{R}^q$ via the action map $\Psi(g, z), g \in G, z \in Z$. All actions in this paper are taken to be regular and so in particular will have orbits of locally constant dimension. The infinitesimal group action, i.e. the induced map of the Lie algebra $\mathbf{g}$ into TZ, the tangent bundle of Z, is written $\psi : \mathbf{v} \rightarrow \psi(\mathbf{v})$. We will usually use "$r$" for the dimension over $\mathbb{R}$ of the algebra of vector fields $\psi(\mathbf{g})$. A set of vectors $\mathbf{S}$ can generate a vector space $< \mathbf{S} >$ over $\mathbb{R}$, or a Lie algebra $\mathbf{a} < \mathbf{R} >$, over $\mathbb{R}$. A p-dimensional regular submanifold of Z will be denoted Γ, especially as when $\Gamma = \Gamma_f$, the graph of a function $f : X \rightarrow U$. All equations considered are locally solvable, so Lie's algorithm generates the full symmetry group $G(\Delta)$. All results are local, referring to a sufficiently small open subset of the manifold, or submanifold, in question.

Definition 2.1. A submanifold $\Gamma \subseteq Z$ is said to be
(i) G **invariant**, if $g \cdot \Gamma \subseteq \Gamma$ for all $g \in G$; and
(ii) G **partially invariant**, if there exists a non-trivial subgroup $H \leq G$ under which Γ is H invariant.

Definition 2.2. For $\Gamma \subseteq Z, G(\Gamma) = \{g \in G : g \cdot \Gamma \subseteq \Gamma\}$ and $\mathbf{g}(\Gamma) = \{\mathbf{v} \in \mathbf{g} : \Gamma$ is flow-invariant under $\psi(\mathbf{v})\}$. Note that $\mathbf{g}(\Gamma)$ is a subalgebra of $\mathbf{g}$ and is the Lie algebra of $G(\Gamma)$.

Remark. The submanifold $\Gamma \subseteq Z$ is G partially invariant if and only if there exists $\mathbf{v} \in \mathbf{g}(\Gamma)$ s.t. $\mathbf{v} \neq 0$.

Terminology. When Γ is partially invariant w.r.t. G, then suppressing the notation for the action Ψ, it is also said that the pair (Γ, G) is **reducible**. Otherwise, the pair (Γ, G) is said to be **irreducible**.

Example 2.3. Let $Z = \mathbb{R} \times \mathbb{R}, \mathbf{g} =< \partial_x, \partial_u >$ and Γ be the x-axis. Then (Γ, G) is reducible using the subgroup generated by $\mathbf{h} = \mathbf{a} < \partial_x >$.

Example 2.4. As above, but let $\mathbf{g} = \mathbf{a} < \partial_u >$. Then (Γ, G) is irreducible.

Definition 2.5. Let $\Delta = 0$ represent a system of nth order differential equations. (See e.g. Olver [5, p.99], where the definition is given for Euclidean space. A more general definition will not be needed for this paper). A diffeomorphism $g : Z \rightarrow Z$ is **a symmetry** of $\Delta = 0$ if g preserves the form of the equation.

From this definition it follows that if f is a solution to $\Delta = 0$ and g is a symmetry, then the transformed function $\tilde{f} = g \cdot f$ is also a solution. Note that $g \cdot f$ is determined from the image of the graph of f under the map $(x, u) \rightarrow g \cdot (x, u)$. (See Olver [5, p.92]). For example, if $Z = \mathbb{R} \times \mathbb{R}$ and $g_\epsilon : (x, u) \to (x + \epsilon, u)$, then a function $f : x \to u = f(x)$ will be mapped to $\tilde{f} : x \to f(x - \epsilon)$.

Definition 2.6. A symmetry group G of $\Delta = 0$ is any Lie group acting on Z which consists of symmetries of $\Delta = 0$; and $G(\Delta)$ represents the maximal such group, when it exists.

We now review certain results from multi-variable calculus. The basic fact used is just that if a function is identically zero on a neighbourhood, then so are all of its derivatives. Using the contrapositive one obtains the standard undergraduate proof that the monomials $1, x, x^2, \ldots$ are linearly independent. In much the same way, if a solution is reducible, then certain linear combinations will equate to zero. Using the contrapositive here a test for irreducibility is obtained.

Let $f : \mathbb{R}^p \to \mathbb{R}$ and suppose that Γ_f is reducible with respect to G (so $\exists \mathbf{v} \in \mathbf{g}$ such that $\mathbf{w} = \psi(\mathbf{v})$ and Γ_f is (flow) invariant under $\psi(\mathbf{v})$).

Let $F(x, u) = u - f(x)$. Then $\Gamma_f = \{(x, u) : F(x, u) = 0\}, \mathbf{w}(z) \in T\Gamma(z)$ and $\mathbf{w}(F)(z) = (gradF)(z) \cdot \mathbf{w}(z) = 0$, for all $z \in \Gamma$. Hence, for all $x \in \mathcal{O}, \mathcal{O}$ open in X, $\mathbf{w}(F)(x, u) = gradF(x, u) \cdot \mathbf{w}(x, u) = 0, u = f(x)$. If $\mathbf{w}_1, \ldots, \mathbf{w}_r$ is a set of generators for $\psi(\mathbf{g})$ then there exist real numbers $\lambda^1, \ldots, \lambda^r$ such that $\mathbf{w} = \lambda^1 \mathbf{w}_1 + \cdots + \lambda^r \mathbf{w}_r$. So we obtain $\left(\sum_{j=1}^{r} \lambda^j \mathbf{w}_j\right)(F) = \sum_{j=1}^{r} \lambda^j \mathbf{w}_j(F) = 0, x \in \mathcal{O}, u = f(x)$. This can be written

$$[\mathbf{w}_1(F) \cdots \mathbf{w}_r(F)] \begin{bmatrix} \lambda^1 \\ \vdots \\ \lambda^r \end{bmatrix} = 0, \quad x \in \mathcal{O}, u = f(x).$$

More generally, when $f : \mathbb{R}^p \to \mathbb{R}^q, f(x) = (f^1(x), \ldots, f^q(x))$ consider

$$F(x, u) = (u^1 - f^1(x), \ldots, u^q - f^q(x)) = (F^1(x, u), \ldots, F^q(x, u)).$$

In this case, when Γ is reducible, obtain $\mathbf{w} = \lambda^1 \mathbf{w}_1 + \cdots + \lambda^r \mathbf{w}_r$ such that

$$\mathbf{w}(F^1) = 0, \ldots, \mathbf{w}(F^q) = 0, \quad x \in \mathcal{O}, u = f(x).$$

In both Olver [5, p.118] and Ondich [6], the quantity $\mathbf{w}(u - f(x))$ is called the "characteristic" of the vector field $\mathbf{w}$ (when $f(x)$ is real-valued) and is denoted by $Q = Q(x, u; f; \mathbf{w})$. Using their notation (see Ondich [6], Sect. 3.1) which applies also to sequences $\mathbf{w}_1, \ldots, \mathbf{w}_r$, our last equation becomes a matrix equation

$$Q(x,u) \begin{bmatrix} \lambda^1 \\ \vdots \\ \lambda^r \end{bmatrix} = \begin{bmatrix} \mathbf{w}_1(F^1) \cdots \mathbf{w}_r(F^1) \\ \vdots \\ \mathbf{w}_1(F^q) \cdots \mathbf{w}_r(F^q) \end{bmatrix} \begin{bmatrix} \lambda^1 \\ \vdots \\ \lambda^r \end{bmatrix} = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix},$$

or

$$\begin{aligned} Q(x,u) \cdot \boldsymbol{\lambda} &= [\mathbf{w}_j(F^i)]\boldsymbol{\lambda} \\ &= [w_j(u^i - f^i(x))]\boldsymbol{\lambda}, \quad i = 1, \ldots, q, \; j = 1, \ldots, r \\ &= 0, \quad x \in \mathcal{O}, u = f(x). \end{aligned}$$

Taking partial derivatives, we obtain

$$\sum_{j=1}^{r} \lambda^j \frac{\partial}{\partial x^k}[\mathbf{w}_j(F^i(x,u))] = 0, \quad k = 1, \ldots, p.$$

Letting $Q^1, \ldots, Q^q$ be the rows of Q and defining

$$\partial_x Q = \begin{bmatrix} \frac{\partial}{\partial x^1} Q^1 \\ \frac{\partial}{\partial x^2} Q^1 \\ \vdots \\ \frac{\partial}{\partial x^p} Q^1 \\ \frac{\partial}{\partial x^1} Q^2 \\ \frac{\partial}{\partial x^2} Q^2 \\ \vdots \\ \frac{\partial}{\partial x^p} Q^q \end{bmatrix}$$

it follows that

$$\partial_x Q \cdot \boldsymbol{\lambda} = 0.$$

Iterating, there follows a sequence of matrices $\partial_x^n Q$, $n = 0, 1, 2, 3, \ldots$ such that

$$\partial_x^n Q \cdot \boldsymbol{\lambda} = 0, \quad x \in \mathcal{O}, u = f(x),$$

where for each n, $\partial_x^n Q$ is a matrix of nth order partial derivatives in $x^1, \ldots, x^p$ of $\mathbf{w}_j(F^i)$, $i = 1, \ldots, q$, $j = 1, \ldots, r$. We record this result as

Proposition 2.7. *Suppose that $f : X \to U$ is reducible with respect to G. Then with $Q(x,u)$ defined as above, there exist real numbers $\lambda^1, \ldots, \lambda^r, \mathbf{w} \in \psi(\mathbf{g}), \boldsymbol{\lambda} = [\lambda^1, \ldots, \lambda^r]^t$ such that $Q \cdot \boldsymbol{\lambda} = 0, \partial_x Q \cdot \boldsymbol{\lambda} = 0, \partial_x^2 Q \cdot \boldsymbol{\lambda} = 0, \ldots$ for $x \in \mathcal{O}$ (some open set in X) and $u = f(x)$.*

Example 2.8. Consider the O.D.E. $y'' = e^{-y'}$. The generators for the Lie symmetries $G(\Delta)$ are given by $X_1 = \frac{\partial}{\partial x}, X_2 = \frac{\partial}{\partial y}, X_3 = x\frac{\partial}{\partial x} + (x+y)\frac{\partial}{\partial y}$. The general solution to the equation is $y(x) = \int \ln(x + D)dx$, where D is constant. A calculation shows that

$$Q(x) = [-\ln(x+D), 1, -x\ln(x+D) + x + \int \ln(x+D)dx].$$

But $\partial_x Q(x) = \left[\frac{-1}{x+D}, 0, \frac{D}{x+D}\right]$.

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If a solution $y = f(x)$ is reducible, then there exist real numbers $\lambda^1, \lambda^2, \lambda^3$ such that $\partial_x Q \cdot \boldsymbol{\lambda} = 0$. This implies $\lambda^1 = \lambda^3 D$ and λ^2 is arbitrary. But if $X = \lambda^1 X_1 + \lambda^2 X_2 + \lambda^3 X_3$ leaves Γ_f invariant, then X is parallel to $(1, f'(x))$, i.e. there exists a function $\mu(x)$ s.t. $X|_\Gamma = \mu \cdot (1, f')$. A calculation now shows that $\lambda^1 = \lambda^2 = \lambda^3 = 0$. Hence every solution is irreducible w.r.t. the full symmetry group $G(\Delta)$.

Proposition 2.9. *Suppose that in the same situation as in Proposition 2.7, there exists a* $\boldsymbol{\lambda}$ *such that* $Q(x,u) \cdot \boldsymbol{\lambda} = 0$, *for all* (x,u) *in some open set* Ω *which contains an open set of the submanifold* Γ_f. *Then* Γ_f *is invariant under the flow given by* $\mathbf{w} = \lambda^1 \mathbf{w}_1 + \cdots + \lambda^r \mathbf{w}_r$. *If* $\mathbf{w} \neq 0$, *then* Γ_f *is reducible.*

Proof. It is known that if $\mathbf{w} \cdot gradF(x,u) = 0$ for all (x,u) in an open set $\Omega \subseteq X \times U$, then by the Mean Value Theorem, restricted to Ω, the flows of $\mathbf{w}$ therefore leave each level set invariant. In particular, the 0-set Γ_f is left invariant.

The problem of finding general criteria for flow invariance has been studied in Optimization Theory and related fields. For example, Clarke [1] gave a result valid for certain Lipschitz functions.

3. CONCLUSIONS

In this paper we give necessary conditions for a solution to be reducible with respect to a given symmetry group G with Lie algebra $\mathbf{g}$. By enlarging the domain of the associated matrix Q, we also give sufficient conditions. The definition of partial invariance, as originally given by Ovsiannikov, was in terms of a defect function δ. Under suitable conditions δ can be given as $\delta(\Gamma, G) = dimG\Gamma - dim\Gamma$. This quantity is related to Q by $\delta \geq rkQ$. We note that the inequality can be strict. In our paper in progress we treat partial invariance in terms of Lie algebra structures. We also relate our results to the work of Ovsiannikov [7], Martina and Winternitz [3], Martina et al [4] and Ondich [6].

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