Introduction to the Supplement

The main text was on vector calculus. Our intention is that this Supplement ~~will~~ be of wider application. The purpose is to help teachers and students make beginnings in describing (instances of your own) mathematical development. As will become evident, this will be useful for all levels of pedagogy.

Part of the novelty of our approach ~~here~~ is that, through examples and directed questions, we invite you to *grow in understanding your own mathematical understanding and that way provide yourself with essentials for growing as a teacher.*

The meaning of the italicized statement will emerge and grow by doing exercises such as those provided in Part A. What we are referring to will be a personal achievement. For the teacher, however, it is rarely an altogether private achievement. We say that because, of course, as teachers, what we think about mathematical development (which includes one's own mathematical development) factors into what we attempt to promote in our students.

Thought on teaching and learning mathematics goes back to antiquity. For instance, although in a different context and not called "mathematics education," there were Plato's reflections about the slave-boy who is helped to solve the problem of doubling the area of a square (*Meno*). However, the discipline now called "mathematics education" is mainly a 20th century development.[41] Good work has been done. For, instance, advances have been

[41] Alan H. Schoenfeld, "Research in Mathematics Education," *Review of Research in Mathematics Education* 40, issue 1 (2016): 497-528. Philip S. Jones, ed., *A History of Mathematics Education in the United States and Canada*, Thirty-Second Yearbook. National Council of Teachers of Mathematics. Reston, VA: National Council of Teachers of Mathematics, 1970 (second printing 2002).

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made in "experiential learning" and "engaged learning." However, looking at the entire discipline, results have been mixed and opinion varies regarding the extent to which methods have been effective, or not.[42] In Part C, we provide a few comments regarding these issues.

What we would like to do first, however (in Part A), is invite you to a few relatively novel[43] exercises in mathematics. There is no "conceptual model" or "representational system." Nor do we draw on or appeal to student test scores, classroom observations, student task orientation, student cooperation, student behavior or other evidence that mathematical learning might or might not be happening for *someone else*. It's not that such results will not eventually contribute to progress in understanding mathematical development in history. The focus that we invite here is more elementary. We invite you to "go to source." The mathematics will be familiar. Part of what is new is that you are invited to a precise "puzzling about your own puzzling in mathematics." The focus, then, is *you*, and *me*, or rather, at least initially, *you-about-you* and *me-about-me*. The invitation is to make initial progress in being able to *advert* to,[44] focus on, inquire about and discern orderings of distinct events in our own inquiry and understanding, in instances, when we are doing mathematics.[45]

We introduce a key diagram (Figure A.1) which could—in a sense—be said

[42]Dawn Leslie and Heather Mendick, eds., *Debates in Mathematics Education*. Abingdon, Oxon: Routledge, 2014.

[43]The approach is not original but will be new for the science of mathematics pedagogy. An advanced source is: Bernard Lonergan, *Insight: A Study of Human Understanding*, eds. Frederick E. Crowe and Robert M. Doran, vol. 3 in The Collected Works of Bernard Lonergan. Toronto: University of Toronto Press, 1992. Introductory level works are: (1) John Benton and Terrance Quinn, *Journeyism*, 2018, `https://bentonfuturology.com/journeyism/`; (2) John Benton, Alessandra Drage and Philip McShane, *Introducing Critical Thinking*. Halifax, Canada: Axial Publishing, 2005 (Reprint 2006). (This book has been translated into Spanish (Madrid: Plaza y Valdés, 2011); and (3) Philip McShane, *Wealth of Self and Wealth of Nations: Self-Axis of the Great Ascent*, Hicksville, NY: Exposition Press, 1975. (This is available for free at `http://www.philipmcshane.org/wp-content/themes/philip/online_publications/books/wealth.pdf`.) Compared to the present Supplement, references (1), (2) and (3) are far broader in their coverage. The Supplement focuses on mathematical development and pushes into further examples.

At the time of posting this preprint, we have learned that a new edition of McShane's book Wealth of Self and Wealth of Nations will be published by Axial Publishing, in 2021.

[44]The (intransitive) verb 'advert' is convenient. We are using it in its first meaning in the Merrian-Webster: "advert, *intransitive verb*. 1: to turn the mind or attention — used with to."

[45]Occasionally, we find it convenient to use a single expression to refer to the whole inquiry: "self-attention." This is meant in the precise sense that will begin to emerge by doing the exercises in Part A. The turn "to attend to one's own understanding" can seem strange if one has been trained to focus mainly on models.

to be a "model." But it is not a "conceptual model" or "speculative framework." Think more of something analogous to what the physics community has been getting to, namely, a "best-to-date standard model" which, for the most part, was discovered in and continues to be verified (or not) ***in instances***, ***in experience***. Or again, in chemistry, one could say that the periodic table is a "model." However, the periodic table emerged from and has been established through centuries of ongoing experimental work investigating chemical dynamics. In a somewhat similar way, we think that you will begin to see that Figure A.1 also is no mere model. By turning attention to our own experience, in exercises in Part A, Figure A.1 emerges and is verified. It too is a "table of elements," that is, "elements of dynamics of knowing." Note that Part A is only a first few "experiments." We leave it to interested readers to go further, to begin exploring the significance of Figure A.1 in other instances in your own mathematical development (and more).[46]

What all of this may have to do with improving teaching in mathematics will be touched on briefly in part B. Part C draws attention to a few anomalies in contemporary mathematics education—in both content and method. A challenge for the education community will be to take advantage of good work that has been done but to also make progress in getting a handle on the various anomalies and misdirects. To do so effectively will not be easy. It will need global collaboration. For this Supplement, it will be enough to tease a few key issues "into view."

[46]See references in note 42.

PART A

Mathematical Understanding

A.1 A diagram

Much in the way an introductory level chemistry text may include a simplified periodic table in the front cover, we begin by providing a simplified version[47]of a key diagram, a "table of elements of knowing." The diagram will be developed as we go.

[47]A more nuanced version is available in Bernard J. F. Lonergan, "Appendix A, Two Diagrams," *in Phenomenology and Logic: The Boston College Lectures on Mathematical Logic and Existentialism*, eds. Frederick E. Crowe, Robert M. Doran, vol. 18 in The Collected Works of Bernard Lonergan, Toronto: University of Toronto Press, 2001, 319–323. A diagram for the "Dynamics of Doing" is the second diagram in Lonergan, "Appendix A, Two Diagrams." Evidently, our "dynamics of doing" are similar to our dynamics of knowing. For decision, however, inquiry is in a different mode. Our dynamics of doing subsumes our dynamics of knowing. The interweaving of the two modes of inquiry are brought out in student exercises in John Benton and Terrance Quinn, "The Dynamics of Doing," *Journeyism* 16 `https://bentonfuturology.com/journeyism16/` and *Journeyism 17*, `https://bentonfuturology.com/journeyism17/`. The dynamics of doing are not an immediate focus of this Supplement. However, it is worth noting that progress in adverting to and describing dynamics of doing also is needed. Among other things, such progress will help resolve numerous contemporary issues in mathematics education where, so far, dynamics of the two modes are not adverted to and consequently, not yet adequately distinguished. For instance, it is sometimes said that a teacher's job is to help a student "decide" which concept or solution to accept. The plausibility of such notions emerges from the mistaken model that mathematical understanding is a matter of connecting concepts. The fact that that view is mistaken is revealed—becomes (self-) evident—by doing the exercises given throughout Part A. For context in mathematics education, see Part C.

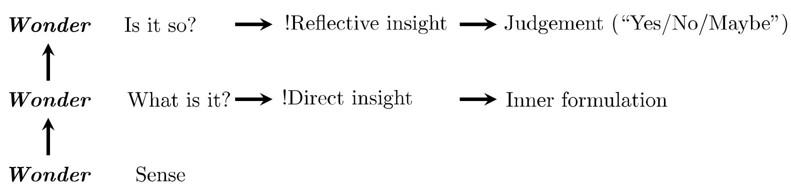


Figure A.1 Dynamics of knowing

A.2 Some puzzles

A.2.1 *A first puzzle*

A first puzzle is given by the following:

1 4 7 11 14 17 41 44 ...

2 3 5 6 8 9 10 12 13 15 16 18 19 20 ... 40 42 43

An ellipsis "$\cdots$" means that the pattern is to be continued. An insight is needed. Let's concentrate on the second ellipsis, the "$\cdots$" that follows '44.'

What?-ing and direct insight

Before getting too far into solving the puzzle, let's pause in a first effort to notice something about what we are doing.

Are you wondering about "the array"? 'What is it?', where the 'it' is the array that you have in sight.

If Yes, then you are "What?-ing", you are in what we could call a "What is it? *inquiry-poise*." Why do I say "*inquiry-poise*" and not "inquiry"? Of course, both are correct. It is for emphasis only that we are adjusting familiar vocabulary somewhat. One wonders. In other words, 'what?-ing' is a transitive verb, something that we do and what?-ing is a holistic poise.

Again, we are not suggesting that we necessarily utter words such as 'What is it?'. In wondering about a seen sequence of symbols we are focused on symbols. Symbols are in and of our senses, are in and of our *sense-ability*. The temporary neologism, then, is to help draw your attention, your self-attention, to the fact that inquiry is inquiry of a whole person and where, in the present example, focus is on something in-and-of one's sense-ability.

Writing down what we have just described, we get part of Figure A.1:

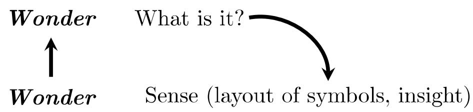


Figure A.2 Wonder about data of sense.

Perhaps you have already discovered a (possible) solution and can now continue the sequence.

If so, something happened. There was a change "in" you. A name for that "all-of-you-event" is *insight*.

Let's call this a *direct insight.*

Why do we need the adjective 'direct'? Partly, it is to distinguish this insight from what turns out to be a second kind of insight that can emerge in a follow-up inquiry-poise discussed below.[48]

For this sequence-puzzle, if you've had a direct insight, then you've gotten hold of something, a possibility.

From that (act of) insight, automatically as it were—but with no implication here of machinery or technology—there is a "procession" in you.

Why do we say "procession"? Focusing on an image in one's sense-ability, being "lit up" by insight "into" an image in one's sense-ability, not only does a light "go on" in us in that concrete image-focus, but light immediately also "goes on in us." There is in us, as it were, "light from light" in the sense that following insight, a solution "emerges in us." There is the famous story of Archimedes, his calling out "Eureka!" As the Greek verb says, when we discover the solution to the sequence puzzle, it is not just an insight. Forthwith, and forth and with, there is also an "I've got *it.*" We are not playing a grammar game here but are inviting your attention to an event. There is an 'it' in "I've got it." There is something that is not merely what we had in our sense-ability. There is something that emerges "in us" from our insight-into-image, something that in many respects is "transferable." For instance, following direct insight, we can turn our attention elsewhere but then later return to the sequence-puzzle. The same solution then comes to mind but without labor.

Why is it without further labor? Because we already get the point. We already "got it." It will be convenient to introduce a name. For historical

[48]See below: Is?-ing and reflective insight.

reasons, let's call the "something that emerges in us," the "it" of "I've got *it*," "inner formulation."[49]

Note that the adjective "inner" is not to suggest that what we get hold of is somehow "spatially inside" (versus a "spatially outside"). The adjective is merely a metaphor. Inner formulation is a fact. It is what we have in as much as we have, in fact, discovered a possible solution. Note that inner formulation is to be distinguished from "formulation" in the sense of providing a "formula." Although, evidently, in some cases, a formula can precede insight, and also be an expression of inner formulation.

An expression for what we have just described is given by the bottom two rows of Figure A.1:

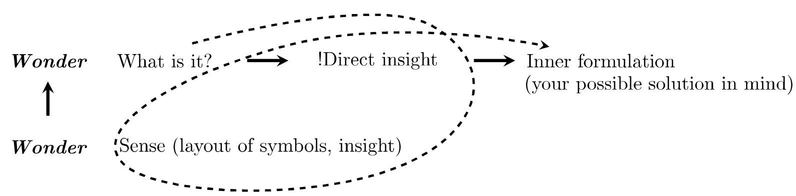


Figure A.3 Wonder about data[50] of sense, insight and inner formulation

Is?-ing and reflective insight

Does the diagram in Figure A.3 express elements that emerge in our inquiry-poise?

Asking that question reveals that in addition to wondering 'What is it?' we also sometimes ask a further question. That is, we also wonder 'Is *it* so?', where the 'it' of 'Is *it* so?' is, again, our possible solution, our inner formulation that emerged in us from our What is it? inquiry-poise.

To be sure, our Is it so? inquiry-poise does not always emerge! Whether because of haste or temperament or circumstances, sometimes we don't get to 'Is it so?' But is it not (self-)[51] evident that we do—at least sometimes—rise to an 'Is it so?' inquiry poise? Is it not so?

[49]Lonergan, *Phenomenology and Logic*, Dynamics of Knowing, "Appendix A," 322. See also McShane, *Wealth of Self*, 15.

[50]See, e.g., Benton and Quinn, "The Dynamics of Knowing: The First Four Boxes," `https://bentonfuturology.com/journeyism10/`; and McShane, Benton, Drage and McShane, ch. 16, "What-Questions," 67-69.

[51]See note 45.

As we find in experience, then, in addition to a 'What is it?' inquiry-poise we also sometimes engage in a further inquiry-poise. We have gotten hold of something through direct insight. We might also reflect on our (potential) resolution to our What is it? inquiry-poise. To be in keeping with tradition, let's call this further inquiry *reflective inquiry*.

Are direct and reflective inquiry so different?

At a dinner party, a colleague asked a mutual friend, "What would you like to drink?" For fun, she answered, "Yes." "Is that so?" She answered, "Tea." It was funny at the time.

The purpose of our story is to help bring out that it is (self-) evident that wondering 'What is it?' and wondering 'Is it so?' are (radically[52]) different questions.

In the sequence puzzle, de facto, the question 'What is that pattern?' is not resolved by answering 'Yes.'

Nor do we resolve an 'Is it so?' question by providing a possible solution to the puzzle.

In other words, it is not merely Yes or No or Probably Yes, or Probably No. We move to an inner Yes (or inner No) *regarding a possible solution* already obtained through direct insight.

In an 'Is it so?' inquiry-poise, what is it that we are we looking for? Continuing with the sequence puzzle, one might, for instance, re-examine the pattern in one's sense-ability, to confirm whether or not one's solution holds up, that is, whether or not one can indeed account for all terms of the sequence. Or, perhaps you feel that you were distracted at the time and don't trust your solution. You might then give the whole sequence-puzzle a fresh look. And so on.

And so on? Notice that our 'Is it so?' inquiry-poise is open ended. You have discovered a possible solution. You have that solution in mind. But 'Is it so?' All that you have done so far is up for grabs. Does your solution hold up under scrutiny? Were you hasty and do you need to start over? Eventually, if one is to not simply guess, something more is needed. What is that "more" that is needed? At what point are we "good to go" (or not) with our possible solution?

In an 'Is it so?' inquiry-poise, we may have another insight. We may reach an understanding to the effect that: Yes, (probably) I have enough to go

[52] [illegible] using the first meaning of the word 'radical,' referring to 'root.'

We are

on. "We reach an understanding that (probably) we have sufficient grounds to assent to our solution." (Note that in this context the word 'probably' is an adjective, not a term from empirical probability theory. Why introduce the adverb? Again, reflection is open ended. We can always ask more questions.)

This can be as spontaneous and rapid as hearing sounds and "immediately" knowing that children are playing soccer in a field across the road. But in mathematics, modern rigor asks us to, as much as possible, spell out what precisely we understand and to identify grounds for what we understand.[53]

In the case of the sequence puzzle the pattern is what we might call a "pattern in symbolism as symbolism."[54] In attempting to resolve our 'Is it so?' inquiry, we again can advert to sense-ability. Now, though, it is not adverting to an image in a 'What is it?' inquiry-poise. Our new lean is to discern whether or not our already discovered solution accounts for all terms of the puzzle. It is a personal achievement: *one* returns to one's puzzle to sort out whether or not *one* can now, in fact, account for all terms of the puzzle.

If you are starting to become somewhat familiar with the more complete focus (that is, of not only doing mathematics but pausing to discern what you are doing when you are doing mathematics), then you may have already noticed that following reflective insight there is a second procession. The second procession is to an inner event called "judgement," an inner "*Yes*, it is so" (or "*No*, it is not so," or *Probably Yes*, or *Probably No*, where, again, in this context, 'probably' is an adverb for the quality of one's judgment). As in procession that follows direct insight, following reflective insight, procession is "automatic" as it were—it occurs. Following reflective insight, procession is to a distinct (but not separable) inner event. In some cultures, occurrence of assent (or its opposite) is obvious because it is spontaneously revealed by a familiar movement of the chin. However, becoming acquainted with people from different parts of the world soon reveals that what that movement is varies among cultures.

Until now, we have held back the solution to the sequence puzzle A.2.1. We did not want to deprive you of the pleasure of discovery.

The sequence $1, 2, 3, \cdots$ is partitioned. Some of the numbers are above a line and some are below. Why? Perhaps surprisingly, a clue is that

[53] That is not easy. See, e.g., note 69.

[54] If you haven't already solved the puzzle, that is a clue. See also the personal anecdote at the end of this section.

problem can be solved by school children. In case you haven't yet solved the problem, we provide the answer below. But in case you wish to keep working on it, we put the solution in a footnote to make it easier to not peek.[55]

Yes, children can solve the problem. But we are inviting you to do something more, something that children are not asked to do (at least not yet, at this time in history). We are inviting you to make beginnings in discerning aspects of what you are doing, what you have done, what you have achieved. You might notice, for instance, that in this case the key insight is "understanding differences in symbols as symbols," that is, getting hold of a pattern in one's sense-ability. If we were to write the ("natural") numbers in Roman Numerals, base 2 or (like the Babylonians) base 60, this puzzle would be destroyed.

There are, however, sequence puzzles that call for other kinds of insight.

A.2.2 *A Famous Sequence*

There is the famous sequence, $1, 1, 2, 3, 5, 8, \cdots$.[56]

If you have already gotten hold of the pattern, notice that, by contrast with the puzzle given in Section A.2.1, what is key here is not understanding what symbols happen to look like, "as symbols."[57] And so, for instance, essentially the same puzzle can be expressed in Roman numerals, base 2 and base 60.

In Roman Numerals, the puzzle can be written as: I, I, II, III, V, VIII, XIII, ...

In a base 2, the puzzle can be written as: $1, 1, 10, 11, 101, 1000, 1101, \cdots$.

[55] Answer to sequence puzzle A.2.1: Symbols that are written only with straight edges go above the line, while symbols that include curved edges go below the line.

[56] The Fibonacci sequence is a sequence of numbers of fertile pairs of male and female rabbits. Start with the hypotheses given by Fibonacci (about fertile male and female pairs, gestation periods and the numbers of progeny). The problem is within reach of the contemporary senior high school or undergraduate student. It is interesting and well worth doing. Discovering the solution for oneself one takes a step in a climb toward the modern theory of recursive sequences.

[57] Long before Fibonacci, the same sequence is found in ancient Sanskrit texts, written in a base 60 number system. See, e.g., Tia Ghose, "What is the Fibonacci Sequence?" *LiveScience* (October 24, 2018): `https://www.livescience.com/37470-fibonacci-sequence.html`; and Keith Devlin, *Finding Fibonacci: The Quest to Rediscover the Forgotten Mathematical Genius Who Changed the World.* Princeton, NJ: Princeton University Press, 2017.

If you have already studied some of the early history of mathematics, you will know that the sequence could also be expressed in cuneiform, using base 60.

An understanding of the sequence can also be expressed implicitly with visibly different formulas, three examples of which are:

$$x_{n+2} = x_{n+1} + x_n,\ n = 0, 1, 2, \cdots; \tag{A.1}$$

$$f_k = f_{k-1} + f_{k-2},\ k = 2, 3, 4, \cdots;\ \text{and} \tag{A.2}$$

$$\Re_{m+20} = \Re_{m+19} + \Re_{m+18},\ m = -18, -17, -16, \cdots.^{58} \tag{A.3}$$

A.3 Descriptive and explanatory understanding, and judgment in mathematics

Descriptive and explanatory understanding

In example (A.1), our understanding is *descriptive*. For the sequence-puzzle A.2.1, *descriptive* refers to the fact that what we grasp is "a pattern in our sense-ability."

By contrast, in example (A.2), our understanding is *explanatory*. Rather than "a pattern in sense-ability," the key insight needed is to discover a pattern of mutually defined terms, and operations.[59]

As is evident from these two examples, both *descriptive* and *explanatory* understanding are part of and contribute to our mathematical development.

For another example, you may recall a familiar sequence from calculus that is often called the 'power rule':

$y = x,\ y' = 1;\ y = x^2,\ y' = 2x;\ y = x^3,\ y' = 3x^2;$ and so on.

[58] From a "higher viewpoint," we could also identify the solution as follows: "The sequence in (A.2) is the (unique) solution to a second-order homogeneous recurrence relation with constant coefficients whose (non-reduced) characteristic equation is $t^2 - t^2 - 1 = 0$, and whose first two initial-values are 1 and 1." See Section A.6.

[59] More generally, descriptive understanding is "grasping patterns in experience, as experience." And so to discern events and orderings of events in our dynamics of knowing is a beginning but it, too, is descriptive. Progress in explanatory understanding of our dynamics of knowing will be future growth for the academic community. Aspects of that future progress are already partly in evidence. And so, eventually, it will include an "integration" of human biophysics, biochemistry, human zoology and psychology, cognitional theory and more. To glimpse something of the challenge, precise densely expressed heuristics can be found in Lonergan, *Insight*, 489 (add the word 'self' to the paragraph that begins "[(Self-) S]tudy of an organism begins ...").

The symbolic pattern can be grasped without needing to understand ~~anything at all about~~ differentiation. In that case, what is grasped is a "symbolic pattern as a symbolic pattern," "a pattern in our sense-ability." But the mathematical meaning of these symbols is reached through explanatory understanding.

Evidently, the observation applies generally. We can understand how to use symbolism in linear algebra; differential equations; and on into the most remote realms of contemporary mathematics. In each case, descriptive understanding is an achievement and by it we understand, for instance, how to use symbols and diagrams. However, there is also the possibility of understanding what symbols and diagrams mean which, in mathematics, is explanatory understanding.[60]

Judgment

Note that in (A.1) and (A.2) we find the same core pattern expressed by Figure A.1. We begin with inquiry about an "image" in sense-ability, and "a focus inquiry." In both cases, a 'What is it?' inquiry-poise is resolved through direct insight, following from which there is a procession "in us" to "inner formulation."

At the same time, there are significant differences in what is being understood. As you might expect, grounds for judgment also turn out to be different.

In example (A.1), direct insight is of a "pattern grasped in our sense-ability." Is our solution correct? Is it so? By adverting again to patterns in our sense-ability, we find and discover sufficient grounds (or not) for assenting to our possible solution. There is then a procession to "Yes, it is so," or perhaps, "No it is not so."

In example (A.2), however, what we discover is a pattern of mutually defined terms, and operations that, as it happens, can be both presented and expressed in many ways.[61] In that case, sufficient grounds for assenting to our possible solution are not "patterns in sense-ability" but whether or not one can, indeed, successfully continue the sequence by appealing

[60]For a discussion in elementary contexts, see Terrance J. Quinn, "On Two Types of Learning (in mathematics) and Implications for Teaching," *On Learning Problems in Mathematics, Research Council on Mathematics Learning*, Mathematical Association of America, FOCUS, Fall 2004, 31–43.

[61]See, e.g., note 60.

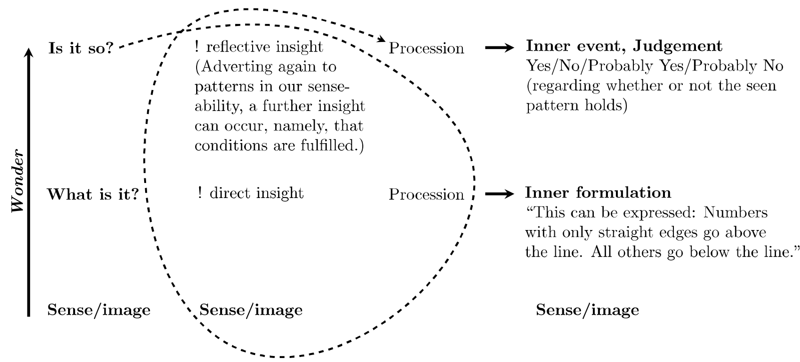


Figure A.4 Reflective inquiry and reflective insight in elementary sequence puzzle A.2.1

to our directly grasped (possible) pattern of mutually defined terms, and operations.

A.4 Descriptive definition and explanatory definition

Let's revisit the 'power rule' sequence: "$y = x,\ y' = 1$; $y = x^2,\ y' = 2x$; $y = x^3,\ y' = 3x^2$; and so on."

In Section A.3, we invited you to observe that in mathematics we enjoy at least two main "genera" of direct insight namely, descriptive and explanatory. Through both, we reach a possible resolution of a 'What is it?' inquiry-poise. But for the power rule, there is only sequence. How do these two resolutions compare? What are we getting at with two answers regarding one sequence?

All along, here, it is a matter of describing what we are doing, of "catching oneself in the act of doing mathematics." From descriptive insight, one understands how to continue the sequence of symbols $y = x,\ y' = 1$; $y = x^2,\ y' = 2x$; $y = x^3,\ y' = 3x^2$. But what do the symbols mean? As in Section A.3, a different understanding is possible. For the power rule sequence, one may also reach a basic understanding of "derivative." [62]

[62] In calculus, there are two key *basic* insights, *basic* in the sense of prior to axiomatics. A satisfactory definition of *limit* did not emerge for several decades after the initial discoveries of calculus. But both Newton and Leibniz each had those two key insights

By descriptive insight, then, we understand how to use symbols and diagrams. This makes it possible to "define" symbols and diagrams in the same that way words are "defined" in a typical dictionary. In other words, thanks to descriptive understanding, we can "define" symbols and other mathematical words and diagrams by indicating how to use them. Naturally enough, we call this *nominal definition*.

By explanatory insight, however, we go on to say what the mathematical symbols, words and diagrams mean. So, there is also *explanatory definition*.

We end this section with one more example. To keep the Supplement short, we leave each sentence as an exercise: Defining an ellipse as a "perfectly symmetrical oval" is a nominal definition. Explanatory definition can be reached from the clue that an ellipse is "a circle with two centers." (It will help to make a diagram for drawing an ellipse. This can include an imagined string, two imagined pins and one imagined pencil or pen.) The key insight needed can lead one to (explanatorily) define an *ellipse* to be the locus of all coplanar points satisfying the equation $F_1P + F_2P = constant$. If one has yet not had an insight into "one's image of a two-center circle," then one's "definition" $F_1P+F_2P = constant$ is nominal. Note, however, that in that case nominal definition is at some remove from an imagined "perfectly symmetrical oval" named 'ellipse.' For, in that case, nominal understanding is not of how to use 'ellipse' as a name for an imagined shape but of how to use the name 'ellipse' in conjunction with a (combination of) symbols '$F_1P + F_2P = constant$.'

A.5 Proofs

My beginnings in calculus, by Terrance Quinn

In September of 1980, in the first weeks of my first-year calculus course at the University of Toronto, the lecturer[63] wrote a theorem on the blackboard about continuity of a function $y = f(x)$. He also provided a proof that went

(and, of course, much more). Once shared with the scientific communities, those two key insights soon made possible the solution of problems from antiquity and also opened up vast ranges of new lines of inquiry and development in mathematical sciences and engineering. See Terrance J. Quinn, "Getting Started in Calculus," *Problems, Resources and Issues in Mathematics Undergraduate Studies* (PRIMUS), vol. 13, issue 1 (March 2003): 55–74.

[63] I am referring to Prof. Edward Bierstone who, as I understand it, is still at the University of Toronto. See `https://www.math.toronto.edu/bierston/`. Accessed July 17, 2019.

something like this: "Suppose that $\epsilon > 0$ and let $\delta = \frac{\epsilon}{4}$. $\cdots$" He went on to fill two boards with inequalities and ended with words similar to: "$\cdots$ from which we can conclude that $|f(x) - f(y)| < \epsilon$. Since $\epsilon > 0$ was arbitrary, the result follows. ■"[64]

I trusted our excellent teacher and had no doubt that "the result *did* follow." But during that lecture I also knew that I didn't yet see why. That evening, I started in on trying to unpack the proof. I found some of the inequalities within the proof ("steps" in the proof) to be a bit tricky. Before too long, however, I could follow the proof "step by step," inequality by inequality. I could see that, yes, each step of the proof worked. But I remember that it was also obvious to me that I still didn't really have it. What I mean is, I was aware that even though I could check all of the steps, I didn't see how I might have produced a similar proof myself. Up to that point, in my mathematical education, I had been naively thinking that I already had a good understanding of calculus. In some respects, I did. However, the first weeks of that first semester of calculus were a shock to me and—as I learned—for some (not all) of my classmates. Eventually, along with other "survivors," I started to find my way. For me, the "epsilon-delta business" crystallized in a problem that involved a damped sine function defined over the real line. In hindsight, I see now that my insight in that case was similar to Archimedes' insight regarding the remainder term of a series. Problem by problem and insight by insight, I crawled my way into modern axiomatics for calculus, a.k.a. "elementary real analysis."[65]

In that first-year class, a common type of problem was of the form: "Prove (or disprove) X (where X was some statement)." How to begin? I soon learned that, for me, unless it was "easily done" (in other words, unless I already understood), it was often best to *not* start with the general statement. It usually worked better for me—that is, I worked better – by starting with examples. Of course, I kept the X in mind and oscillated back and forth between examples and thinking about the X statement. But inevitably, the break for me would come while working on an example. That started to take some the mystery out of "proof writing." In other words, "being able to write a proof is not separate from understanding."

I noticed that the order in which the write-up proceeded usually turned out

[64]The textbook in the 1980–81 academic year was the edition then available of Michael Spivak, *Calculus*. Cambridge: Cambridge University Press.

[65]Historically, axiomatics for calculus were obtained after several decades of collaboration following the initial breakthroughs of Newton and Leibniz.

to be, to some extent, in a kind of reverse order to what I might call "the order of inquiry and discovery." I now realize that was no accident. I have been recounting some of my early struggles in a challenging and exciting first-year calculus course. But to begin homing in on key and core issues, it is better to now shift to a more elementary example, essential features of which are already described by another author.

Understanding and syllogism

> In a circle of, say, unit radius, two diameters, perpendicular to each other, are drawn. [You will need to make a diagram.] From an arbitrary point P on the circumference two perpendiculars PR and PS are drawn to the two diameters. The problem is, What is the ratio of RS to the radius? ... Joining R and S will be an evident thing to do; but it may take a pedagogue to adequately dispose the phantasm by the drawing of another line. The line to draw is the line joining the center to the point P, say OP. Eureka! With the insight there emerges the solution, the relation between RS and the radius.
>
> Note now that the solution can be formulated or thrown into syllogistic form, and this will help you get some light on features of the syllogism that are often misrepresented. We have, therefore, the syllogism:

	$RS = \text{OP}$
and	OP = Radius;
therefore	RS = Radius.

In this light we may note important characteristics of the procedure. We started not with two premises, but with the conclusion in the form

RS ? Radius.

Our search, through diagram, was for a middle term, and the middle term was supplied as soon as one adverted to the significance of OP. Only then can the syllogism be constructed. To coin an expression for this constructing, one might say that the insight is crystallized into a syllogism. This does not mean of, of course, that somehow insight has been pinned down on a page. What has happened is that we have given the insight symbolic expression. Giving all the relevant insights symbolic expression is by no means always an easy thing to do, even when it can be done. Modern geometers have found fault with Euclid in this matter. There are insights in the *Elements* which are not explicitly acknowledged either in axioms or in the theorems, yet which were not uncrystallizable.[66]

Let me note further that our simple puzzle and solution is a paradigm of how Euclid and company may well have proceeded. They did not proceed step by step down the page of a modern textbook, from the stated theorem to the fully constructed diagram, to the step by step deduction beneath.[67]

We may return now to a brief consideration of the simple symbolic expression of understanding which is the (mathematical) syllogism.

[In mathematics, the syllogism is a] help toward understanding.

The syllogism is not some mysterious replacement for understanding.

One may look at the syllogism as a proof of the conclusion, but this can only mean that the structure facilitates a grasp of the implication of the conclusion in the premises.

One might note, too, that such structures facilitate the checking, the Is-question, relating to that grasp.[68]

[66]It concerns drawing a line across the interior of a triangle. One identifies the point where the line crosses the opposite side. Why is there a point of intersection? Felix Klein discusses the problem: "Of course, no one would doubt that, intuitively, but in the framework of axiomatic deduction we need a special axiom, the so-called betweenness axiom for the plane. This axiom states that if a line enters a triangle through a side, it must leave it through the other side—a trivial fact of our space perception which requires emphasis as such, because it is independent of the other axioms. . . . If we omit this, as Euclid does, we cannot reach the ideal of a pure logical control of geometry. We must continually refer to the figure" (Felix Klein, *Elementary Mathematics from an Advanced Standpoint: Geometry* (Mineola, NY: Dover Publications, 2004), 88. This is an unabridged republication of an earlier Dover reprint (1949) of the translation first published by The Macmillan Company, New York, in 1939). As is well known, there are non-Euclidean geometries for which the axiom does not hold. For further discussion, see McShane, *Wealth of Self*, 68-9.

[67]"One should note, too, the significance of this for the teaching of geometry. Too often pupils [are asked to] begin, not with the thrill of a puzzle but with the top of the page, and at most get a vague line-by-line comprehension of the theorem. Memory is burdened, and examinations consists in filling out theorems - from the bottom and top of the page!—and passing over the riders" (McShane, *Wealth of Self*, 69). This provides foreshadowing to Part B.

[68]Philip McShane, *Wealth of Self*, 67–9.

A.6 Algebra from arithmetic, "and so on": that is, sequences of "higher viewpoints"

When were children, we learned to count.

"Five baskets of apples and 7 baskets of apples is 12 baskets of apples."

Before long, however, through insight, we learned to 'abstract.'[69]

For instance, in this example, one might go on to understand that whatever we happen to be counting, we can count "how many":

$7 \cdot 3$'s $+ 5 \cdot 3$'s $= (7+5) \cdot 3$'s $= 12 \cdot 3$'s; and

$5 \cdot 9$'s $+ 8 \cdot 9$'s $= (5+8) \cdot 9$'s $= 13 \cdot 9$'s.

Abstracting further,

$7 \cdot 3 + 5 \cdot 3 = (7+5) \cdot 3 = 12 \cdot 3$; and

$5 \cdot 9 + 8 \cdot 9 = (5+8) \cdot 9 = 13 \cdot 9$;

and so on.

And so on?

By insight we are beginning to transition "beyond" arithmetic.

One can solve many arithmetic problems without breaking into, or up into, algebra. That is both historical and biographical. In moving into algebra, however, we do not lose arithmetic. We still understand, for instance, that $5 + 7 = 12$. But now we can also write and understand that $5z + 7z = (5+7)z = 12z$, whatever the number z happens to be.

By insight, one understands and holds together an indefinite range of instances in arithmetic.

Moving further into the new context:

$5 \cdot 3 + 7 \cdot 3 = (5+7) \cdot 3$

$7 \cdot 3 + 5 \cdot 3 = (7+5) \cdot 3 = 12 \cdot 3$

$5 \cdot 9 + 8 \cdot 9 = (5+8) \cdot 9 = 13 \cdot 9$

$\ldots$

?

(direct insight)

from which one can write: $x \cdot z + y \cdot z = (x+y) \cdot z$.

[69]Here, 'abstract' is a transitive verb for what we do in thought: "*abstracten*" to draw away, remove, derivative of *abstract* (or borrowed directly from Latin *abstractus*)" `https://www.merriam-webster.com/dictionary/abstract`.

As discussed in Section A.3, that insight may be nominal, as when a student grasps "a pattern in symbolism as symbolism." But one might also understand that x, y and z represent any possible numbers, and that the expression $x \cdot z + y \cdot z = (x + y) \cdot z$ represents a pattern of operations. In that case, we not only know how to use the symbols x, y and z, we also have gotten hold of a correlation of mutually defined terms, and operations, by which one can explain what the expression means. Moreover, with that understanding, we can now just as easily write $a \cdot c + b \cdot c = (a + b) \cdot c$, $U \cdot V + W \cdot V = (U + W) \cdot V$, (U times V) plus (W times V) = (U plus W) times V, and so on.

Transitioning from arithmetic to algebra is sometimes called transitioning to a "higher viewpoint." In the case of arithmetic and algebra, both "viewpoints" are *systems*. The word 'higher' is a spatial metaphor, but it serves nicely. For, as revealed in instances, by understanding in algebra one has control over indefinite ranges of arithmetic computations.

A key point: Self-notice that images by which we reach the higher viewpoint of algebra are diagrams and symbols from lower viewpoints such as arithmetic and elementary geometry.

We are not asserting this key point as a conclusion from a "theoretical framework" or model for mathematical development. It is an observation reached by adverting to instances to one's own experience.[70]

In mathematics, does transitioning from lower to higher viewpoints happen often?

In order to make beginnings toward being able to answer this Is-question one would need to have (considerable) experience in mathematics. One would also need to attend to what one has been doing. The question calls for and depends on growth in mathematics as well as growth in the kind of "balanced" attention being invited in this Supplement. In other words, there is the further challenge of adverting to, noticing and discerning as precisely as possible what one is doing when one is doing mathematics.[71]

[70] As you might already be seeing, this has implications for teaching. See Part B.
[71] The task is further described in Section A.8.

A.7 Correlations, concepts and other fruit of understanding

Correlations

In coordinate (Cartesian) geometry, the "parent parabola" is defined by the equation $y = x^2$. Other parabolas are obtained by adjusting the scale of one or both of the axes, by translation, and by rotation.

We start with this definition because it will be familiar to many teachers. It is what is often given in high school and first year undergraduate books.

The present challenge is to make a beginning in "unpacking" (what turn out to be "layerings" of) insights that often are neither expressed nor adverted to.

Let's begin with our focus on $y = x^2$, an algebraic formula. In that context, what are the x and the y? By using a pair of symbols, we refer to an indefinite range of pairs of numbers, some of which are (0, 0), (-1, 1), (1, 1), (-2, 4), (2, 4), (-3, 9), (3, 9), and so on. In each pair, what is x? It represents every number that squares to give a corresponding number y. What is y? It represents every number that is the square of some x. The x and the y are mutually defined by a correlation that can be expressed by '$y = x^2$'. Note that in this context, terms, operations and equality are algebraic. But in that correlating, we also understand and correlate an indefinite range of pairs of numbers where in each case, terms, operations and equality are arithmetic.

That's a start. However, we haven't finished "unpacking." Why not? Coordinate geometry is not merely algebra. In coordinate geometry, the x and the y are coordinates. In other words, the x and y refer to lengths along axes that, by hypothesis, have scales. So, let's look more closely to our underlying (or perhaps latent) understanding of an x-axis.

One can imagine a distinguished straight line in the Euclidean plane[72] and call it the 'x-axis.' In coordinate geometry, we also have a scale along that distinguished line (and, indeed, along all lines in the same Euclidean plane).

[72] What is a *straight line* in Euclidean geometry? (Here, we avoid the extensive literature regarding "primitive concepts," "primitive," and "vicious circles in logic." Such problems need to be handled, but not here.) In Book I of the *Elements*, Euclid provides what evidently are nominal definitions: "Def. 1.2. *A line is a breadthless length.*" "Def. 1.4. *A straight line lies equally with respect to the points on itself.*" Explanatory definition is obtained when a straight angle is defined to be the sum of two 90-degree angles. In that way, we go beyond merely describing what we imagine. In that case, a straight angle is defined by correlating terms in the system.

And so we can write $x = 0$, $x = 1$, $x = 2$, $x = 22/7$, $x = e$, $x = \pi$, and so on. In coordinate geometry, these numbers refer to lengths along the distinguished axis.

This may seem obvious enough. But is it?

Let's take one of these, $x = 2$, say. In this context, to what do we refer with the number 2?

Here, $x = 2$ is a "distance" along the x-axis.

How is that distance determined?

You might remember that the task here is not to invent something new but to discern elements in what we already do. Distance is given in terms of a *unit distance* along the x-axis.

OK, but what is a *unit distance*?

In physical measurement, a *unit distance* is whatever happens to be convenient. This might be the length of one's thumb (an inch, or in French, 'un pouce,' which means 'thumb'), a handy length of wood, the distance from the tip of one's middle finger to one's elbow (cubit), a "standardized cubit" such as the length of a piece copper used by the Sumerians in the third millennium B.C.E., and so on. In modern times, a 'meter' is the length of a bar of platinum held in Paris, France.

In mathematics, distance is not a physical length. Still, there is nothing to stop us from imagining and thinking about a unit length,

" —— ".

Physical lengths can and do change. Two people with different arm lengths will yield different cubits. A piece of wood may dry out and its length may then change. Or again, suppose we have two metal rulers with "cms" etched along their sides. In other words, "1 cm" is the unit length. If one of the rulers is used at different locations on a hot wood stove then "1 cm" at one location can be longer than "1 cm" at another, when compared with the second ruler that is kept from direct contact with the hot stove. But if there is no fire in the stove, and if no finer measurements are needed then, for practical purposes, no problems arise. That is, the unit length is "invariant." Through the ages, builders, carpenters, architects and engineers have relied on such invariance when designing and building structures—from wooden shacks to pharaohs' pyramids, from modern homes to city skyscrapers.

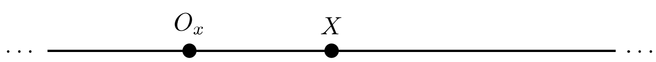
In classical geometry, we “take this idea and run with it.” We suppose that an imagined unit length is invariant in the sense that an imagined unit length at one location is the same as an imagined unit length at another.

Let’s now return our focus to the x-axis. Let O_x be a (distinguished) point on the x-axis. The point O_x is called “origin.” Let X be another point on the same axis, a point that is, say, ‘two units’ “to the right” of O_x. Notice our ongoing reliance on imagination, e.g., “to the right,” and “to the left.”

We have an imagined x-axis which is an imagined distinguished line in the Euclidean plane. By insight, we suppose that the line extends indefinitely in both directions:

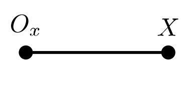
“... ———————————————— ...”

We imagine two points on this imagined x-axis, O_x and X:



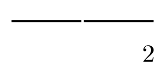
In the imagined length between these two imagined locations, one can imagine fitting copies of the imagined unit length, “ ——— ” and “ ——— ”, distinct line segments that, by hypothesis, are otherwise identical. In particular, we suppose that they have the same length.

A handy diagram for this is:

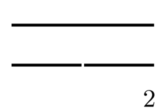


So, in our understanding, we correlate an imagined line segment with copies of an imagined unit length:

But we also correlate an imagined “ ——— ——— ” with the number ‘2’, a diagram for which is:



In coordinate geometry, “$x = 2$” means that we take both of these correlations “together,” a diagram for which is:



In other words, "together" means that we need a further insight by which we obtain, grasp, or reach a further correlation. That is, we correlate two correlations.

What we find then is that, in coordinate geometry, asserting that $x = 2$ is an expression of an insight by which one holds together a "layering" of insights: one correlates two correlations. Similarly, $x = 3$ is an expression for correlating two correlations; and so on for all values of x; and similarly for all values of y.

We started with the expression $y = x^2$. If we only focus on numbers, then our understanding is an algebraic correlation. In coordinate geometry, however, the x's and y's are lengths along two distinguished (and perpendicular) lines in the Euclidean plane, a plane in which there is, by hypothesis, an invariant unit length. Unpacking key insights, the x-values represent an aggregate of correlations of correlations; as do the y-values. In *coordinate geometry*, then, $y = x^2$ is an expression of a *correlation of correlations of correlations.*

Concepts

In Euclidean geometry, we speak of 'points,' 'lines,' 'line segments,' 'length,' 'invariance,' and so on, as well as the entire 'Euclidean plane.'

What are these?

What, for instance, is a *point*?

We can imagine a small dot on paper (or, say, on a computer screen). An imagined dot has breadth and so we can imagine two smaller dots within an imagined dot. Still, "the point" is "clear," is it not? That is, attempting to "pin down" a location unambiguously, we need to keep making an imagined dot smaller. So long as a dot has any breadth or depth, further dots can be imagined within the imagined dot.

By **insight**, then, we define a *point* on the plane to be "a location that has no breadth and no depth."

Or, in translation from Book I of Euclid's *Elements: "A point is that which has no part."*

A *point* then, **(self-) evidently** is the fruit of insight! It is also a concept.[73]

[73] "concept: noun, (Entry 1 of 2) 1: something conceived in the mind: thought, notion, 2: an abstract or generic idea generalized from particular instances" `https://www.merriam-webster.com/dictionary/concept`.

In a similar way, that is, by adverting to experience, one can find that lines, planes, invariant length, and so on, are also the fruit of insight and also are concepts.

Continuing in this way, what becomes (self-) evident is that while it is true that images and patterns in our sense-ability on which we focus inquiry can be said to be "primitive," points, lines and other concepts are the fruit of understanding and, in particular, are neither reducible to imagination nor are they primitive elements in mathematics.

A.8 The historical context for teachers (and scholars)

How do we go on from elementary and preliminary exercises? It is for each teacher and scholar in mathematics to work out how far you need to go. "Just as in any subject, one masters the essentials by varying the incidentals."[74] In so far as one has the time and the proclivity, one can identify key diagrams and symbolic expressions, helpfully directed questions, key insights, skills, and sequences of such in one's own mathematical development.

You might observe that, unless one is breaking new ground, when understanding mathematics one is also sharing in key insights from the historical development of the field. Note that this is not a statement regarding "the historical method" for teaching. This is an *observation* about learning mathematics, for instance, algebra, or calculus. How else is one to know and teach "completing the square" that was discovered in various times and places in antiquity, or "the calculus" that was discovered by Newton and Leibniz, and know these in ways that allow one to teach them effectively, unless one has not (at least partially) identified key questions and insights? At the same time, (self-) evidently, it is always our own inquiry and our own insights that we experience. Our own development is our main source of data[75] on mathematical development.

It was Bernard Lonergan who first brought attention to the need and possibility of the balanced method to which we are inviting mathematics teachers. And so we end Part A of this Supplement with a quotation from Lonergan on historical understanding [of mathematical development].

[74]Lonergan, *Insight*, 56.
[75]See note 50.

The original quotation is from a discussion of historical understanding of a discipline.

> The history of [mathematics] is in fact the history of its development. But this development, which would be the theme of a history [of mathematics], is not something simple and straightforward but something that has occurred in a long series of various stages, errors, detours, and corrections. To the extent that the one studying this movement learns about this developmental process, one already possesses within oneself an instance of that development which took place perhaps over several centuries. This can happen only if one understands both the subject and the way in which he or she learned about it. Only then will one understand which elements in the historical developmental process had to be understood before others, which were the causes of progress in understanding and which held it back, which elements really belong to that particular science and which do not, and which elements contained errors. Only then will one be able to tell at what point in the history of the subject there emerged new visions of the whole and the first true system occurred, and when transitions took place from an earlier to a later systematic ordering; which systematization was simply an expansion of the former and which was radically new; what progressive transformations the whole subject matter underwent; how everything that was previously explained by the old systematization is now also explained by the new one, as well as many other things that the old one did not explain [as in the discoveries in mathematics, for example, by Euclid, Archimedes, Apollonius, Pappus, Al-Khwarizmi, Descartes, Fermat, Newton, Leibniz, Cauchy, Fourier, Riemann, Galois, the great analysts, and others]. Only then will one be able to understand what factors favored progress, what hindered it, why, and so forth.
>
> Clearly, therefore, [master teachers of mathematics] have to have a thorough knowledge and understanding of the whole subject. And it is not enough that they understand it in any way at all, but they must have a systematic understanding of it. For the precept, when applied to [the history of mathematics], means that

they must understand successive systems that have progressively developed over time. This systematic understanding of a development ought to make use of an analogy with the development that takes place in the mind of [a teacher] who is learning the subject, and this interior development within the mind of the [teacher] ought to parallel the historical process by which [mathematics] itself developed.[76]

[76]Bernard Lonergan, Early *Works on Theological Method 2*, vol. 23 in the *Collected Works of Bernard Lonergan*, translated by Michael G. Shields, edited by Robert M. Doran and H. Daniel Monsour (Toronto: University of Toronto Press, 2013): 175–177. For a somewhat dated but still excellent source on mathematical development, see, Eric Temple Bell, *The Development of Mathematics.* New York: McGraw–Hill, 1940. Second Edition: New York, McGraw–Hill, 1945. Reprint: Dover Publications, 1992. University of Toronto Press, 2013): 175–177. For a somewhat dated but still excellent source on mathematical development, see, Eric Temple Bell, *The Development of Mathematics.* (New York: McGraw–Hill, 1940. Second Edition: New York, McGraw–Hill, 1945. Reprint: Dover Publications, 1992).

PART B

A Few Implications for Teaching

It is for brevity that we merely state a few results. To bring these out pedagogically would need a few chapters. However, we hope that providing these few results will serve as a guide and invitation to further reflection. If you have worked through the exercises in Part A (or similar ones), you may get to some (or all) of these yourself. As pointed to in Section A.8, further details and implications will be discovered through one's ongoing growth in mathematics and in self-attention in mathematics.[77] We do not comment on curricula of any departments or colleges involved in the ongoing challenging and creative work of meeting students' needs and program needs in diverse circumstances.

Part of the invitation is for a teacher to make progress in identifying and distinguishing nominal and explanatory understandings, in one's own understanding.

As experience reveals, nominal understanding includes understanding by which one reaches competence with symbolic techniques and computations.

The task of pedagogy invites teachers to know as much as possible about where, in precise terms, the content of a course fits in the historical development of the field. See Section A.8.

By the same token, in order to be able to help a student make progress in the field, it will help if a teacher learns as much as possible about where a student's understanding is in relation to the historical development of the field.

As expressed by Figure A.1, by adverting to one's experience in doing mathematics, it becomes evident that understanding emerges through inquiry

[77]See note 45. "balanced method."

"into" images in one's sense-ability. Consequently, in a given mathematical context, it will help if a teacher is familiar with diagrams and symbolisms that help raise and direct student inquiry toward reaching key insights appropriate to a context.

To help a student reach a higher viewpoint, it will help if a teacher can provide diagrams and symbolisms from lower viewpoints by which a student (or group of students) can reach key insights needed to begin breaking into the higher viewpoint. See Section A.6.

For a century and more, there has been ongoing discussion about whether or not teachers should "use history" to teach mathematics. An advantage of the "historical method" is that it has the potential for helping students raise questions and reach key insights in the field. But of course not all of the searchings from history can be included. A selection needs to be made and tailored to an audience and to a curriculum. In that sense, a teacher needs to "make history better than it was." More precisely, a teacher can make progress in identifying specific developmental sequences.[78] In that way, for instance, in two leisurely 45 minute classroom sessions one can help senior high school mathematics students and freshmen level general calculus students reach the two key insights had by both Newton and Leibniz from which the entire body of calculus is developed.[79]

Pedagogy called engaged learning has been a fruitful idea. The importance and relevance of modern psychology and neuroscience (so, e.g., behaviour, environmental considerations, learning style, group work, and so on) to pedagogy cannot be denied. But might we not also ask, engaged in what? We are back at the need and possibility of identifying helpfully directed diagrams and symbols by which to subtly guide student inquiry toward specific and key mathematical insights.

Only a small percentage of students go on to become mathematics majors who, in addition to needing *basic*[80] key insights, also go on to (sequences of) axiomatic systems. But students in applied mathematics, engineering, applications and general service courses also need to be helped so that inquiry be directed toward the emergence of initial or basic key insights. Why? Without basic insights (which, as experience reveals, precede the emergence of axiomatic system), understanding is mainly nominal. Even if a student is only wanting to be able to do routine computations in a

[78] See note 76.
[79] See note 62.
[80] See note 62.

practical career, if understanding is merely nominal, they will be unable to proceed whenever boundary conditions do not sufficiently mesh with what is only nominally familiar.

Modern textbooks speak of "understanding concepts" and "conceptual understanding."[81] It is said (or assumed) that "concepts are primitive elements" and that mathematical understanding is a matter of "connecting concepts" (that one has prior to understanding). It is also now popular to assert analogies between structures of computer programs and human understanding. There is an 'Is it so?' question here. By adverting to instances in our experience (see, e.g., Section A.7), it is (self-) evident that while there are "primitive elements" in mathematics (for instance, in Euclidean geometry), it is not concepts that are primitive. And the experiential basis of alleged analogies with computer programming are not found in mathematical understanding. They are obtained, rather, by correlating structures of computer programs with structural features of hypothetical conceptual models of mathematical understanding. On the other hand, adverting to experience in mathematical understanding reveals that points, lines, invariant unit length, and other familiar concepts emerge from insight. In other words, concepts are the fruit of understanding.

Relative to arithmetic, algebra is a higher viewpoint. As already described, one reaches a higher viewpoint by abstracting[82] from instances in a lower viewpoint.

This sheds light for us on the problem of giving students calculators and other computational technologies too soon. If a student does not have a good understanding and technical competence in arithmetic, then they are lacking most if not all of the symbolic data, diagrams and understanding needed in order to break through to the higher viewpoint of algebra. Similarly, if a student lacks nominal and explanatory understanding in algebra and coordinate geometry, they will be lacking most if not all experience needed in order to break through to the higher viewpoint of calculus. Again, there are transitions from algebra to abstract algebra, from calculus to function theory, and so on.

Evidently, building up an axiomatic system relies on understanding that is beyond initial or basic understanding (see, e.g., Section A.5). This helps reveal a common misdirect, that is, when a chapter or lesson begins with "preliminary concepts," "axioms" and/or "axiomatically correct

[81]See Part C.
[82]See note 69.

definitions." Certainly, that approach is dull and is a way to undermine engaged learning. Attempting to start with axioms and preliminary concepts is, in fact, attempting to start with answers to questions from a further context. The approach does not bring the student into an inquiry zone that promotes emergence of initial key insights. But this is another face of the misconception in mathematics education that is often called "conceptual understanding."[83]

Whether implicit or explicit, a teacher's view of mathematical development influences how and what we teach. Whatever one's context in teaching, and whether private or shared with colleagues, there is the possibility of reflecting on one's own experience in mathematics and on what one hopes one's students also will learn. In other words, as invited in Part A, there is the possibility of growing in being able to advert to and precisely distinguish elements in one's own experience. If one is already a successful teacher, the invitation is to accurately identify sources of one's success and that way become an even more effective teacher. If one is struggling as a junior teacher, the invitation can include the task of identifying gaps in one's mathematical understanding, inconsistencies in what one understands, and elements of one's own understanding (specific combinations of nominal and explanatory, in and across diverse contexts), and to make efforts to bring one's learning and teaching into better harmony with the historical field and the needs of one's students.

[83]See three paragraphs above. On the other hand, 'conceptual understanding' is just a name. Many excellent teachers use the term in a positive sense, namely, when referring to student understanding that goes beyond mere technical competence. Implicitly, they are referring to both nominal and explanatory understanding in mathematics. The name "conceptual understanding" however is, in fact, inherited from the philosophical literature and in that context refers to what does not occur in mathematics. That is, as experience reveals, we do not understand concepts but, rather, concepts emerge from understanding.

PART C

Observations Regarding Modern Mathematics Education

For decades, emphases in mathematics education have been on "theoretical frameworks."[84] Attempts to justify a theoretical framework are made by working out features of, and in, a framework; designing lessons based on such derivations; and then appealing to observations and statistical analyses of, for instance, student scores, student behavior, reported feelings, and numerous other "aggregate-events" remote to source events that are mathematical inquiry and understanding itself.[85] Unfortunately, this approach has been promoting fundamentally mistaken views of mathematical understanding.

We provide only a few remarks. We are not attempting to give a scholarly discussion of contemporary mathematics education.[86] Our strategy here mainly is to tease at a few strands of the problem. Among other things, this will help teachers begin see that it is possible to personally assess models that one may be asked to implement.

Let's start by looking to the Introduction of the well-known and influential paper by Dubinsky and McDonald (2001).[87]

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[84]Schoenfeld, 2016. For a survey of "theoretical frameworks used in the field of Calculus education," see David Bressoud, I. Ghedamsi, V. Martinez-Luaces, G. Törner, G. *Teaching and Learning of Calculus*, ICME-13 Topical Surveys, Springer Open (2016) 1-37. `https://www.springer.com/gp/book/9783319329741`.

[85]See, e.g., Schoenfeld, 2016.

[86]That will be a community achievement, coming from a new kind of collaboration on a global scale.

[87]E. Dubinsky and M. A. McDonald, "APOS: A constructivist theory of learning in undergraduate mathematics education research," in D. Holton et al. (Eds.), *The Teaching and learning of Mathematics at University Level: An ICMI Study* (Dordrecht, Netherlands: Kluwer Academic Publishers, 2001): 273–280.

> We do not think that a theory of learning is a statement of truth and although it may or may not be an approximation to what is really happening when an individual tries to learn one or another concept in mathematics, this is not our focus. Rather we concentrate on how a theory of learning mathematics can help us understand the learning process by providing explanations of phenomena that we can observe in students who are trying to construct their understandings of mathematical concepts and by suggesting directions for pedagogy that can help in this learning process.[88]

Note that, in the first sentence, the authors express a lack of concern for whether or not their theory is "a statement of truth." They then remove themselves from the task of inquiring into "what is really happening when an individual tries to learn": "this is not our focus." Their focus is, instead, "phenomena that we can observe in students."[89] The last sentence of the paragraph reveals that in their inquiry into learning, they presuppose a theory of learning. That this, they presuppose a version of constructivism which, in this context, is to the effect that students "learn one or another concept in mathematics"; that they "construct their understanding of mathematical concepts"; (later in the paper) that when learning mathematics one "[perceives] mathematical problem situations"; and that, when learning, "an individual is developing her or his understanding of a concept."

If you have done exercises such as those in Part A (or beyond) and have made beginnings in understanding your own understanding in mathematics, do not the claims made in the paper cry out for correction?[90]

Perhaps, though, the resulting pedagogy stands. So, let's look at what Dubinsky and McDonald suggest for teaching *cosets*:

> Pedagogy is then designed to help the students make these mental constructions and relate them to the mathematical con-

[88]Dubinsky and McDonald, 2001.

[89]If one's focus is on "phenomena that we can observe in students" then inquiry will not be about mathematical insight but, rather, patterns in "phenomena what we can observe in students" who do not yet understand. Note, too, that if (as in APOS and its applications) those phenomena are not explanatorily defined, statistical analysis has little or at most preliminary explanatory significance.

[90]Regarding concepts, see Section A.7. The confusion is historical and not unique to the work of Dubinsky and McDonald.

> cept of *coset*. In our work, we have used cooperative learning and implementing mathematical concepts on the computer in a programming language which supports many mathematical constructs in a syntax very similar to standard mathematical notation.

Evens and odds, "clock arithmetic" and other similar arithmetic and geometric groupings are learned by children. In that way, students learn much of what is needed in order to prepare the way for reaching a higher viewpoint that includes an understanding of cosets. In our own mathematical development, we learned mathematics, and did not make use of syntax "similar to standard mathematical notation." Indeed, in the historical development of cosets, founders did not appeal to computer programming language. Is it not evident that syntax (of computer programming) "very similar to standard mathematical notation" is, in fact, a distraction from the *mathematical* problem?

Why did Dubinksy and McDonald advocate that approach to teaching cosets? Why attempt to use something similar to mathematics to teach mathematics instead of helping students learn mathematics itself? Indeed, why might that be relevant (or not) when "what is really happening" is, as they suggested, not the focus and not known? They explain it as follows:

Their design instruction focused, not directly on mathematics, but on some model of how the topic in question can be learned.[91]

Dubinsky and McDonald had a remarkable dedication to the cause of mathematics education. As scientists, however, they chose to not focus on "what is really happening" but rather on "phenomena that we can observe in students." In that approach, they also "took themselves out of the equation" and so did not avail of experience in mathematical understanding.

Constructivist models allegedly account for mathematical understanding. And so, to reveal the presence of flaws, all that is needed is a counterex-

[91] Ed Dubinksy, "Using a Theory of Learning in College Mathematics Courses," *MSOR Connections* (2000), 1.10.11120/msor.2001.01020010. This article was originally published in Newsletter 12, TaLUM, Teaching and Learning Undergraduate Mathematics subgroup, 2001. It is available online: http://www.math.wisc.edu/ wilson/Courses/Math903/UsingAPOS.pdf. Accessed July 17, 2019. The tradition continues. The literature is extensive. For a point of entry into the literature see, e.g., L. Benton, C. Hoyles, I. Kalas and R. Noss, "Designing for learning mathematics through computer programming: A case study of puplis engaging with place value," *International Journal of Child-Computer Interaction*, vol. 16 (2018), 68–76.

ample. But by doing exercises in Part A (or similar ones), one can obtain numerous counterexamples across a range of instances in elementary mathematics.

Concluding comments for Part C

As already alluded to, conceptual models of human understanding are not new. Part of a larger tradition called "conceptualism," roots of contemporary constructivism go back at least as far as claims made by Duns Scotus (1266–1308).[92]

Aspects of engaged learning models are proving to be significant and are partly grounded in progress in human neuroscience and human psychology. But in instances, where a particular mathematical result is needed in what, *specifically*, is a student to be engaged? And so, a model might be partly valid but also call for development. On the other hand, there are "theoretical frameworks" that are mainly speculative.

How does one tell the difference and so make positive use (or not) of models encountered?

One can always ask:

Does the model explain instances, in detail, in *my* experience?

If a model of mathematical learning does not bear out in one's own instances of mathematical learning, then (self-) evidently the model is in some way flawed. Note that what is novel[93] here and most important is not this or that model. Neither are we suggesting that teachers or other readers believe what we have only very briefly touched on this Supplement. Rather, we are inviting teachers to a way by which one can check for oneself and so obtain "ground to stand on," one's own experience. We have found, and ~~for instance,~~ we hope that you are finding, that by adverting to one's own experience (by doing exercises like those given in Part A, and more), one can make progress in identifying and distinguishing orderings of key and core

[92]In Scotus' speculative theory of knowing, "[i]ntellectual intuitive cognition does not require phantasms; the cognized object somehow just causes the intellectual act by which its existence is made present to the intellect. As Robert Pasnau notes, intellectual intuitive cognition is in effect a 'form of extra-sensory perception' (Pasnau [2002])" (Williams, Thomas, "John Duns Scotus," *The Stanford Encyclopedia of Philosophy* (Spring 2016 Edition), Edward N. Zalta (ed.), `https://plato.stanford.edu/archives/spr2016/entries/duns-scotus/`. See `https://plato.stanford.edu/entries/duns-scotus/#MatForBodSou`.

[93]The method is not new but will be new in mathematics education. See note 43.

events in sequences of instances in one's own mathematical development.

Among those who engage in such exercises, there will be differences in mathematical background and in descriptions of elements in one's inquiry and understanding.[94] So, we can expect that there will also be differences in views regarding learning and teaching mathematics. Like in any serious science, such differences will not be resolved by logical debate about models. However, to the best of our ability we can attempt to spell out and discuss aspects of our own experience, inquiry and growth in mathematics.

[94]Although, the basic structure given in Figure A.1 is invariant. If one claims that a different structure accounts for one's understanding, is it because one understands something about an understanding that one does not have; or that one judges what one does not understand? In the terminology of philosophy of science, there results what is called 'performance-contradiction.' Philosophical terminology is not the main issue here. As in the text above, a fundamental question always is: "Does a model explain *instances*, in detail, in *my* experience?" Is this *me*?